AP Calculus BC Notes

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Contents

Chapter 1

Functions and Models

1.1 Four Ways to Represent a Function

Definition 1.1.1. A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

Definition 1.1.2. The set D is called the domain of the function. The set of all possible values of $f(x)$ as x varies throughout the domain is called the range .

Example 1. Sketch the graph and find the domain and range of each function.

(a) $f(x) = 2x - 1$

(b) $g(x) = x^2$

(c) $f(x) = \sqrt{x+2}$

(d) $g(x) = \dfrac{1}{x^2 - x}$

***Proposition* 1** (Vertical Line Test)**:** A curve in the xy-plane is a the graph of a function of x if and only if no vertical line intersects the curve more than once.

Definition 1.1.3. Piecewise defined functions are defined by different formulas in different parts of their domains.

Example 2. A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq -1, \\ x^2 & \text{if } x > -1. \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

Definition 1.1.4. The absolute value of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line.

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

Example 3. Sketch the graph of the absolute value function $f(x) = |x|$.

Definition 1.1.5. If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an <u>even function</u>.

Example 4. The function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x).$$

Definition 1.1.6. If a function f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an <u>odd function</u>.

Example 5. The function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

Example 6. Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$

(b) $g(x) = 1 - x^4$

(c) $h(x) = 2x - x^2$

1.2 Mathematical Models

Definition 1.2.1. y is a linear function of x if the graph of the function is a line. The slope-intercept form of the equation of can be used to write a formula for the function as

$$y = f(x) = mx + b$$

where m is the slope of the line and b is the y-intercept.

Definition 1.2.2. A function P is called a polynomial if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the coefficients of the polynomial. If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n.

Example 1. The function

$$P(x) = 2x^6 - x^4 + \frac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

Remark 1. A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function. A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a quadratic function. A polynomial of degree 3 is of the form $P(x) = ax^3 + bx^2 + cx + d$ and is called a cubic function.

Definition 1.2.3. A function of the form $f(x) = x^a$, where a is a constant, is called a power function. If $a = n$, where n is a positive integer, $f(x) = x^n$ is a polynomial. If $a = 1/n$, where n is a positive integer, $f(x) = x^{1/n} = \sqrt[n]{x}$ is a root function. If $a = -1$, $f(x) = x^{-1} = 1/x$ is a reciprocal function.

Definition 1.2.4. A rational function f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

Example 2. The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$.

Definition 1.2.5. A function f is called an algebraic function if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials.

Example 3. The functions

$$f(x) = \sqrt{x^2+1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x-2)\sqrt[3]{x+1}$$

are algebraic.

Definition 1.2.6. Trigonometric functions are functions of an angle that relate the angles of a triangle to the lengths of its sides.

Remark 2. The sine, cosine, and tangent functions are the most familiar trigonometric functions. The convention in calculus is that radian measure is always used, unless otherwise indicated.

Remark 3. For all values of x, we have

$$-1 \leq \sin x \leq 1 \qquad -1 \leq \cos x \leq 1,$$

or equivalently,

$$|\sin x| \leq 1 \qquad |\cos x| \leq 1.$$

Also, the periodic nature of these functions implies that

$$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$$

for all values of x.

Example 4. What is the domain of the function $f(x) = \dfrac{1}{1 - 2\cos x}$?

Definition 1.2.7. Exponential functions are functions of the form $f(x) = b^x$, where the base b is a positive constant.

Definition 1.2.8. Logarithmic functions are functions of the form $f(x) = \log_b x$, where the base b is a positive constant.

Remark 4. Logarithmic functions are inverse functions of exponential functions.

Example 5. Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \dfrac{1+x}{1-\sqrt{x}}$

(d) $u(t) = 1 - t + 5t^4$

1.3 New Functions from Old Functions

***Proposition* 1** (Vertical and Horizontal Shifts)**:** Suppose $c > 0$. To obtain the graph of
$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

***Proposition* 2** (Vertical and Horizontal Stretching and Reflecting)**:** Suppose $c > 1$. To obtain the graph of
$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Example 1. Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x-2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

Example 2. Sketch the graphs of the following functions.

(a) $y = \sin 2x$

(b) $y = 1 - \sin x$

Example 3. Sketch the graph of the function $y = |x^2 - 1|$.

Definition 1.3.1. The sum and difference functions are defined by

$$(f + g)(x) = f(x) + g(x) \qquad (f - g)(x) = f(x) - g(x).$$

Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0.$$

Definition 1.3.2. Given two functions f and g, the composite function $f \circ g$ (also called the composition of f and g) is defined by

$$(f \circ g)(x) = f(g(x)).$$

Example 4. If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

Example 5. If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$, find each of the following functions and their domains.

(a) $f \circ g$

(b) $g \circ f$

(c) $f \circ f$

(d) $g \circ g$

Example 6. Find $f \circ g \circ h$ if $f(x) = x/(x+1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

Example 7. Given $F(x) = \cos^2(x + 9)$, find functions f, g, and h such that $F = f \circ g \circ h$.

1.4 Exponential Functions

***Proposition* 1** (Laws of Exponents)**:** If a and b are positive numbers and x and y are any real numbers, then

1. $b^{x+y} = b^x b^y$ 2. $b^{x-y} = \dfrac{b^x}{b^y}$ 3. $(b^x)^y = b^{xy}$ 4. $(ab)^x = a^x b^x$

Example 1. Sketch the graph of the function $y = 3 - 2^x$ and determine its domain and range.

Example 2. Use a graphing calculator to compare the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$. Which function grows more quickly when x is large?

Example 3. The half-life of strontium-90, ^{90}Sr, is 25 years. This means that half of any given quantity of ^{90}Sr will disintegrate in 25 years.

(a) If a sample of ^{90}Sr has a mass of 24 mg, find an expression for the mass $m(t)$ that remains after t years.

(b) Find the mass remaining after 40 years, correct to the nearest milligram.

(c) Use a graphing calculator to graph $m(t)$ and use the graph to estimate the time required for the mass to be reduced to 5 mg.

Definition 1.4.1. We call the function $f(x) = e^x$ the natural exponential function where e is the value of b in $y = b^x$ resulting in a tangent line at $(0, 1)$ with slope 1.

Example 4. Graph the function $y = \frac{1}{2}e^{-x} - 1$ and state the domain and range.

Example 5. Use a graphing device to find the values of x for which $e^x > 1,000,000$.

1.5 Inverse Functions and Logarithms

Definition 1.5.1. A function is a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2.$$

***Proposition* 1** (Horizontal Line Test)**:** A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Example 1. Is the function $f(x) = x^3$ one-to-one?

Example 2. Is the function $q(x) = x^2$ one-to-one?

Definition 1.5.2. Let f be a one-to-one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

for any y in B.

Example 3. If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

Remark 1. The letter x is traditionally used as the independent variable, so when we concentrate on f^{-1} we usually reverse the roles of x and y to get

$$f^{-1}(x) = y \Leftrightarrow f(y) = x.$$

By substituting for x and y, we get the following cancellation equations

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$
$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Example 4. Find the inverse function of $f(x) = x^3 + 2$.

Definition 1.5.3. The logarithmic function with base b, denoted by $\log_b$, is the inverse function of the exponential function $f(x) = b^x$ with $b > 0$ and $b \neq 1$, i.e.,

$$\log_b x = y \Leftrightarrow b^y = x.$$

Remark 2. By the cancellation equations,

$$\begin{aligned} \log_b(b^x) &= x \quad \text{for every } x \in \mathbb{R} \\ b^{\log_b x} &= x \quad \text{for every } x > 0. \end{aligned}$$

***Proposition* 2** (Laws of Logarithms)**:** If x and y are positive numbers, then

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b(x^r) = r\log_b x$ (where r is any real number)

Example 5. Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

Definition 1.5.4. The logarithm with base e is called the natural logarithm and is denoted by

$$\log_e x = \ln x.$$

Example 6. Find x if $\ln x = 5$.

Example 7. Solve the equation $e^{5-3x} = 10$.

Example 8. Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

***Proposition* 3:** For any positive number b ($b \neq 1$), we have

$$\log_b x = \frac{\ln x}{\ln b}.$$

Example 9. Evaluate $\log_8 5$ correct to six decimal places.

Example 10. Sketch the graph of the function $y = \ln(x-2) - 1$.

Definition 1.5.5. The inverse sine function or arcsine function, denoted by $\sin^{-1}$, is the inverse of the sine function on the restricted domain $[-\pi/2, \pi/2]$.

Remark 3. By the cancellation equations,

$$\sin^{-1}(\sin x) = x \quad \text{for} \; -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
$$\sin(\sin^{-1} x) = x \quad \text{for} \; -1 \leq x \leq 1.$$

Example 11. Evaluate (a) $\sin^{-1}(\frac{1}{2})$ and (b) $\tan\left(\arcsin\frac{1}{3}\right)$.

Definition 1.5.6. The inverse cosine function or arccosine function, denoted by $\cos^{-1}$, is the inverse of the cosine function on the restricted domain $[0, \pi]$.

Remark 4. By the cancellation equations,

$$\begin{aligned} \cos^{-1}(\cos x) &= x \quad \text{for } 0 \leq x \leq \pi \\ \cos(\cos^{-1} x) &= x \quad \text{for } -1 \leq x \leq 1. \end{aligned}$$

Definition 1.5.7. The inverse tangent function or arctangent function, denoted by $\tan^{-1}$, is the inverse of the tangent function on the restricted domain $[-\pi/2, \pi/2]$.

Example 12. Simplify the expression $\cos(\tan^{-1} x)$.

Chapter 2

Limits and Derivatives

2.1 The Tangent and Velocity Problems

A tangent to a curve is a line that that touches the curve. A secant is a line that cuts a curve more than once.

Example 1. Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Example 2. The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The data in the table describe the charge Q remaining on the capacitor (measured in microcoulombs) at time t (measured in seconds after the flash goes off). Use the data to draw the graph of this function and estimate the slope of the tangent line at the point where $t = 0.04$. [*Note:* The slope of the tangent line represents the electric current flowing from the capacitor to the flash bulb (measured in microamperes).]

t	Q
0.00	100.0
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

Example 3. Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds. [If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law that the distance fallen by any freely falling body is proportional to the square of the time it has been falling is expressed by the equation $s(t) = 4.9t^2$.]

2.2 The Limit of a Function

Definition 2.2.1. Suppose $f(x)$ is defined when x is near the number a. Then we write

$$\lim_{x \to a} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a but not equal to a.

Example 1. Guess the value of $\lim\limits_{x \to 1} \dfrac{x-1}{x^2-1}$.

Example 2. Estimate the value of $\lim\limits_{t \to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$.

Example 3. Guess the value of $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$.

Example 4. Investigate $\lim_{x\to 0} \sin \frac{\pi}{x}$.

Example 5. Find $\lim_{x\to 0} \left(x^3 + \frac{\cos 5x}{10,000} \right)$.

Definition 2.2.2. We write

$$\lim_{x\to a^-} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x less than a. Similarly, if we require that x be greater than a, we write

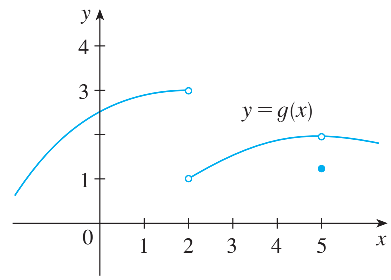
$$\lim_{x\to a^+} f(x) = L.$$

Example 6. Investigate the limit as t approaches 0 of the Heaviside function H, defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

Remark 1. $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

Example 7. Use the graph of g to state the values (if they exist) of the following:



(a) $\lim_{x\to 2^-} g(x)$ (b) $\lim_{x\to 2^+} g(x)$

(c) $\lim_{x\to 2} g(x)$ (d) $\lim_{x\to 5^-} g(x)$

(e) $\lim_{x\to 5^+} g(x)$ (f) $\lim_{x\to 5} g(x)$

Definition 2.2.3. Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a. Similarly,

$$\lim_{x\to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

Example 8. Find $\lim_{x\to 0} \frac{1}{x^2}$ if it exists.

Definition 2.2.4. The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

Example 9. Find $\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\lim_{x\to 3^-} \frac{2x}{x-3}$.

Example 10. Find the vertical asymptotes of $f(x) = \tan x$.

2.3 Calculating Limits Using the Limit Laws

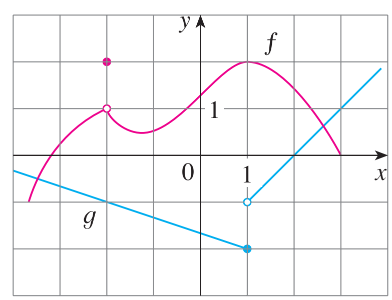
***Proposition* 1** (Limit Laws)**:** Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

Example 1. Use the Limit Laws and the graphs of f and g to evaluate the following limits, if they exist.



(a) $\lim_{x\to -2}[f(x)+5g(x)]$

(b) $\lim_{x\to 1}[f(x)g(x)]$

(c) $\lim_{x\to 2}\dfrac{f(x)}{g(x)}$

***Proposition* 2** (Power and Root Laws)**:** By repeatedly applying the Product Law and using some basic intuition we obtain the following:

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

7. $\lim_{x \to a} c = c$

8. $\lim_{x \to a} x = a$

9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer
    (If n is even, we assume that $a > 0$.)

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer
    $\left[\text{If } n \text{ is even, we assume that } \lim_{x \to a} f(x) > 0.\right]$

Example 2. Evaluate the following limits and justify each step.

(a) $\lim_{x \to 5} (2x^2 - 3x + 4)$

(b) $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

***Proposition* 3** (Direct Substitution Property)**:** If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a).$$

Example 3. Find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

Remark 1. If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

Example 4. Find $\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} x + a & \text{if } x \neq 1, \\ \pi & \text{if } x = 1. \end{cases}$$

Example 5. Evaluate $\lim_{h \to 0} \frac{(3 + h)^2 - 9}{h}$.

Example 6. Find $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$.

Example 7. Show that $\lim_{x\to 0} |x| = 0$.

Example 8. Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Example 9. If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4, \\ 8-2x & \text{if } x < 4. \end{cases}$$

determine whether $\lim_{x\to 4} f(x)$ exists.

Example 10. The greatest integer function is defined by $[\![x]\!]$ = the largest integer that is less than or equal to x. (For instance, $[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}]\!] = 1$, $[\![-\frac{1}{2}]\!] = -1$.) Show that $\lim_{x\to 3}[\![x]\!]$ does not exist.

Theorem 2.3.1. *If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then*

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

Theorem 2.3.2 (The Squeeze Theorem)**.** *If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L.$$

Example 11. Show that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

2.4 The Precise Definition of a Limit

Definition 2.4.1. Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Example 1. Use a graph to find a number δ such that if x is within δ of 1, then $f(x) = x^3 - 5x + 6$ is within 0.2 of 2.

Example 2. Prove that $\lim_{x \to 3}(4x - 5) = 7$.

Definition 2.4.2.

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Similarly,

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Example 3. Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Example 4. Prove that $\lim_{x \to 3} x^2 = 9$.

Definition 2.4.3. Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) > M.$$

Similarly,

$$\lim_{x \to a} f(x) = -\infty$$

means that for every negative number N there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) < N.$$

Example 5. Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

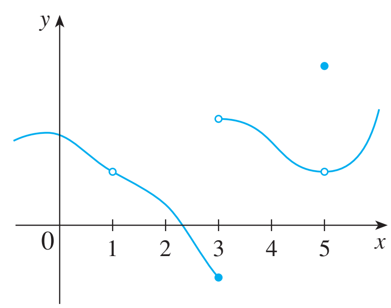
2.5 Continuity

Definition 2.5.1. A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a).$$

We say that f is discontinuous at a (or f has a discontinuity at a) if f is not continuous at a.

Example 1. Use the graph of the function f to determine the numbers at which f is discontinuous.



Example 2. Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

Definition 2.5.2. A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

Example 3. In which direction(s) is the function $f(x) = [\![x]\!]$ continuous?

Definition 2.5.3. A function f is continuous on an interval if it is continuous at every number in the interval.

Example 4. Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

Theorem 2.5.1. *If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:*

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\frac{f}{g}$ *if* $g(a) \neq 0$

Proof. All of these results follow from the Limit Laws. For example, $f + g$ is continuous at a because

$$\begin{aligned}\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\ &= f(a) + g(a) \\ &= (f+g)(a).\end{aligned}$$ □

Theorem 2.5.2. *(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.*

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Proof. (a) Let

$$P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

be a polynomial where $c_0, c_1, \ldots, c_n$ are constants. Then

$$\lim_{x\to a} x^m = a^m \qquad m = 1, 2, \ldots, n$$

implies that the function $f(x) = x^m$ is continuous. Since

$$\lim_{x\to a} c_0 = c_0,$$

the constant function is continuous as well, and therefore the product function $g(x) = cx^m$ is continuous. Since P is a sum of functions of this form, it is continuous as well.

(b) Rational functions are quotients of polynomials, i.e.,

$$f(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials. Thus the above result implies that they are continuous on their domains. □

Example 5. Find $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$.

Theorem 2.5.3. *The following types of functions are continuous at every number in their domains:*

- *polynomials*
- *rational functions*
- *root functions*
- *trigonometric functions*
- *inverse trigonometric functions*
- *exponential functions*
- *logarithmic functions*

Example 6. Where is the function $f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?

Example 7. Evaluate $\lim_{x \to \pi} \frac{\sin x}{2 + \cos x}$.

Theorem 2.5.4. *If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$, i.e.,*

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

Proof. Let ε. Since f is continuous at b, we have $\lim_{y\to b} f(y) = f(b)$ and so there exists $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |y-b| < \delta_1 \qquad \text{then} \qquad |f(y) - f(b)| < \varepsilon.$$

Since $\lim_{x\to a} g(x) = b$, there exists $\delta > 0$ such that

$$\text{if} \quad 0 < |x-a| < \delta \qquad \text{then} \qquad |g(x) - b| < \delta_1.$$

By letting $y = g(x)$ in the first statement, we get that $0 < |x-a| < \delta$ implies that $\big|f(y) - f(b)\big| < \varepsilon$, i.e., $\lim_{x\to a} f(g(x)) = f(b)$. □

Example 8. Evaluate $\lim_{x\to 1} \arcsin\left(\dfrac{1-\sqrt{x}}{1-x}\right)$.

Theorem 2.5.5. *If g is continuous at a and f is continuous at $g(a)$, then the composite function $f\circ g$ given by $(f\circ g)(x) = f(g(x))$ is continuous at a.*

Proof. Since g is continuous at a, we have

$$\lim_{x\to a} g(x) = g(a).$$

Since f is continuous at $g(a)$, we have

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(g(a)),$$

which means $f\circ g$ is continuous. □

Example 9. Where are the following functions continuous?

(a) $h(x) = \sin(x^2)$

(b) $F(x) = \ln(1 + \cos x)$

Theorem 2.5.6 (Intermediate Value Theorem)**.** *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.*

Example 10. Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

2.6 Limits at Infinity

Definition 2.6.1. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Definition 2.6.2. Let f be a function defined on some interval $(-\infty, a)$. Then

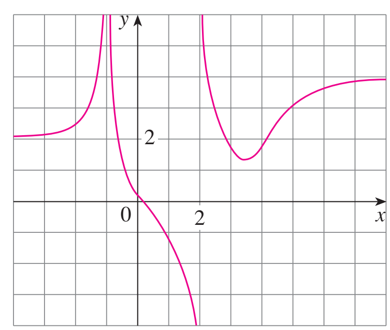
$$\lim_{x\to-\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

Definition 2.6.3. The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

Example 1. Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown.



Example 2. Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.

Theorem 2.6.1. *If $r > 0$ is a rational number, then*

$$\lim_{x \to \infty} \frac{1}{x^r} = 0.$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0.$$

Proof. By extending the limit laws to limits at infinity we get

$$\lim_{x \to \infty} \frac{1}{x^r} = \lim_{x \to \infty} \left[\frac{1}{x}\right]^r = \left[\lim_{x \to \infty} \frac{1}{x}\right]^r = 0^r = 0$$
$$\lim_{x \to -\infty} \frac{1}{x^r} = \lim_{x \to -\infty} \left[\frac{1}{x}\right]^r = \left[\lim_{x \to -\infty} \frac{1}{x}\right]^r = 0^r = 0.$$

□

Example 3. Evaluate

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}.$$

Example 4. Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}.$$

Example 5. Compute $\lim_{x\to\infty}(\sqrt{x^2+1}-x)$.

Example 6. Evaluate $\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right)$.

Example 7. Evaluate $\lim_{x\to 0^-} e^{1/x}$.

Example 8. Evaluate $\lim_{x\to\infty} \sin x$.

Example 9. Find $\lim_{x\to\infty} x^3$ and $\lim_{x\to-\infty} x^3$.

Example 10. Find $\lim_{x\to\infty} (x^2 - x)$.

Example 11. Find $\lim_{x\to\infty} \dfrac{x^2 + x}{3 - x}$.

Example 12. Sketch the graph of $y = (x-2)^4(x+1)^3(x-1)$ by finding its intercepts and its limits as $x \to \infty$ and as $x \to -\infty$.

Definition 2.6.4. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Definition 2.6.5. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x < N \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Example 13. Use a graph to find a number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1.$$

Example 14. Prove that $\displaystyle\lim_{x\to\infty} \frac{1}{x} = 0$.

Definition 2.6.6. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad f(x) > M.$$

Definition 2.6.7. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = -\infty$$

means that for every negative number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad f(x) < M.$$

Definition 2.6.8. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = \infty$$

means that for every positive number M there is a corresponding negative number N such that

$$\text{if} \quad x < N \qquad \text{then} \qquad f(x) > M.$$

Definition 2.6.9. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = -\infty$$

means that for every negative number M there is a corresponding negative number N such that

$$\text{if} \quad x < N \qquad \text{then} \qquad f(x) < M.$$

2.7 Derivatives and Rates of Change

Definition 2.7.1. The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Example 1. Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Example 2. Use the alternative expression for the slope of a tangent line

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

to find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

Definition 2.7.2. A function f describing the motion of an object along a straight line is called a position function and has velocity

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

at time $t = a$.

Example 3. Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground. Recall that the distance (in meters) fallen after t seconds is $4.9t^2$.
(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Definition 2.7.3. The derivative of a function f at a number a, denoted by $f'(a)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

or equivalently

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

Example 4. Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

Example 5. Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

Definition 2.7.4. Suppose y is a quantity that depends on another quantity x. Then y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the increment of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1).$$

The average rate of change of y with respect x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

and the instantaneous rate of change of y with respect to x is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x).$$

Example 6. A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is $C = f(x)$ dollars.
(a) What is the meaning of the derivative of $f'(x)$? What are its units?

(b) In practical terms, what does it mean to say that $f'(1000) = 9$?

(c) Which do you think is greater, $f'(50)$ or $f'(500)$? What about $f'(5000)$?

Example 7. Let $D(t)$ be the US national debt at time t. The table gives approximate values of this function by providing end of year estimates, in billions of dollars, from 1985 to 2010. Interpret and estimate the value of $D'(2000)$.

t	$D(t)$
1985	1945.9
1990	3364.8
1995	4988.7
2000	5662.2
2005	8170.4
2010	$14,025.2$

Source: US Dept. of the Treasury

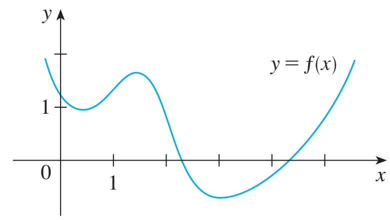
2.8 The Derivative as a Function

Definition 2.8.1. The derivative of a function f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if this limit exists.

Example 1. The graph of a function f is given. Use it to sketch the graph of the derivative f'.



Example 2. (a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.

(b) Illustrate this formula by comparing the graphs of f and f'.

Example 3. If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

Example 4. Find f' if $f(x) = \dfrac{1-x}{2+x}$.

Definition 2.8.2. The symbols D and d/dx are called differentiation operators and are used as follows:

$$f'(x) = y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x).$$

For fixed a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_{x=a}$$

Definition 2.8.3. A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Example 5. Where is the function $f(x) = |x|$ differentiable?

Theorem 2.8.1. *If f is differentiable at a, then f is continuous at a.*

Proof. If f is differentiable at a, we have

$$\begin{aligned}\lim_{x\to a}[f(x)-f(a)] &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}(x-a)\\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a)\\ &= f'(a)\cdot 0 = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}[f(a)+(f(x)-f(a))]\\ &= \lim_{x\to a} f(a) + \lim_{x\to a}[f(a)-f(a)]\\ &= f(a)+0 = f(a).\end{aligned}$$

□

Definition 2.8.4. If the derivative f' of a function f has a derivative of its own we call it the second derivative of f and denote it by

$$(f')' = f'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Example 6. If $f(x) = x^3 - x$, find and interpret $f''(x)$.

Definition 2.8.5. The instantaneous rate of change of velocity with respect to time is called the acceleration $a(t)$ of an object. It is the derivative of the velocity function, and therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t).$$

Definition 2.8.6. The third derivative f''' is the derivative of the second derivative, denoted by

$$(f'')' = f'''.$$

Definition 2.8.7. The instantaneous rate of change of acceleration with respect to time is called the jerk $j(t)$ of an object. It is the derivative of the acceleration function, and therefore the third derivative of the position function:

$$j(t) = a'(t) = v''(t) = s'''(t).$$

Definition 2.8.8. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Example 7. If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.

Chapter 3

Differentiation Rules

3.1 Derivatives of Polynomials and Exponentials

Theorem 3.1.1. *The derivative of a constant function* $f(x) = c$ *is* 0*, i.e.,*

$$\frac{d}{dx}(c) = 0.$$

Proof.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0. \qquad \square$$

Theorem 3.1.2.

$$\frac{d}{dx}(x) = 1 \qquad \frac{d}{dx}(x^2) = 2x \qquad \frac{d}{dx}(x^3) = 3x^2 \qquad \frac{d}{dx}(x^4) = 4x^3$$

Proof. All of these follow directly from the definition of the derivative, as above. □

Theorem 3.1.3 (The Power Rule)**.** *If* n *is a positive integer, then*

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof. Since

$$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}),$$

we have

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^n - a^n}{x-a} \\
&= \lim_{x\to a} (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) \\
&= a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1} \\
&= \underbrace{a^{n-1} + a^{n-1} + \cdots + a^{n-1} + a^{n-1}}_{n} \\
&= na^{n-1}.
\end{aligned}$$

□

Example 1. Find the derivative of each of the following:

(a) $f(x) = x^6$

(b) $y = x^{1000}$

(c) $y = t^4$

(d) $f(r) = r^3$

Theorem 3.1.4 (The Power Rule (General Version))**.** *If n is any real number, then*

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Example 2. Differentiate:

(a) $f(x) = \dfrac{1}{x^2}$

(b) $y = \sqrt[3]{x^2}$

Definition 3.1.1. The <u>normal line</u> to a curve C at a point P is the line through P that is perpendicular to the tangent line at P.

Example 3. Find equations of the tangent line and normal line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$.

Theorem 3.1.5 (The Constant Multiple Rule)**.** *If c is a constant and f is a differentiable function, then*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x).$$

Proof. Let $g(x) = cf(x)$. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} \\ &= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right] \\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= cf'(x). \end{aligned}$$

□

Example 4. Find:

(a) $\dfrac{d}{dx}(3x^4)$

(b) $\dfrac{d}{dx}(-x)$

Theorem 3.1.6 (The Sum Rule)**.** *If f and g are both differentiable, then*

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Proof. Let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\ &= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right] \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x). \end{aligned}$$

□

Theorem 3.1.7 (The Difference Rule)**.** *If f and g are both differentiable, then*

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

Example 5. Find $\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$.

Example 6. Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

Example 7. The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

Definition 3.1.2. e is the number such that $\lim_{h \to 0} \dfrac{e^h - 1}{h} = 1$.

Theorem 3.1.8. $\dfrac{d}{dx}(e^x) = e^x$.

Example 8. If $f(x) = e^x - x$, find f' and f''.

Example 9. At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

3.2 The Product and Quotient Rules

Theorem 3.2.1 (The Product Rule). *If f and g are both differentiable, then*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

Proof. By the definition of the derivative on the product,

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)[g(x+h)-g(x)]}{h} + \lim_{h\to 0}\frac{g(x)[f(x+h)-f(x)]}{h} \\
&= \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]. \qquad \square
\end{aligned}$$

Example 1. (a) If $f(x) = xe^x$, find $f'(x)$.

(b) Find the nth derivative, $f^{(n)}(x)$.

Example 2. Differentiate the function $f(t) = \sqrt{t}(a + bt)$.

Example 3. If $f(x) = \sqrt{x}g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

Theorem 3.2.2 (The Quotient Rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Proof. Similar to the Product Rule, except we add and subtract $f(x)g(x)$ in the numerator when applying the definition of the derivative. □

Example 4. Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Find y'.

Example 5. Find an equation of the tangent line to the curve $y = e^x/(1+x^2)$ at the point $(1, \frac{1}{2}e)$.

3.3 Derivatives of Trigonometric Functions

Theorem 3.3.1. *The derivative of the sine function is the cosine function, i.e.,*

$$\frac{d}{dx}(\sin x) = \cos x.$$

Example 1. Differentiate $y = x^2 \sin x$.

Theorem 3.3.2. *The derivative of the cosine function is the negative sine function, i.e.,*

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Theorem 3.3.3. *The derivative of the tangent function is the square of the secant function, i.e.,*

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

Proof. By the Quotient Rule,

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x.
\end{aligned}$$

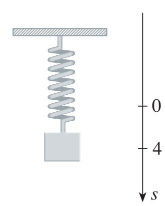
□

Theorem 3.3.4. *The derivatives of the trigonometric functions are*

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\tan x) = sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x.$$

Example 2. Differentiate $f(x) = \dfrac{\sec x}{1+\tan x}$. For what values of x does the graph of f have a horizontal tangent?

Example 3. An object at the end of a vertical spring is stretched to 4 cm beyond its reset position and released at time $t = 0$. (See the figure and note that the downward direction is positive.) Its position at time t is



$$s = f(t) = 4\cos t.$$

Find the velocity and acceleration at time t and use them to analyze the motion of the object.

Example 4. Find the 27th derivative of $\cos x$.

Example 5. Find $\lim_{x\to 0} \frac{\sin 7x}{4x}$.

Example 6. Calculate $\lim_{x\to 0} x \cot x$.

3.4 The Chain Rule

Theorem 3.4.1 (The Chain Rule). *If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product*

$$F'(x) = f'(g(x)) \cdot g'(x).$$

Example 1. Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

Example 2. Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

Theorem 3.4.2 (The Power Rule Combined with the Chain Rule)**.** *If n is any real number and $u = g(x)$ is differentiable, then*

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

Example 3. Differentiate $y = (x^3 - 1)^{100}$.

Example 4. Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

Example 5. Find the derivative of the function

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9.$$

Example 6. Differentiate $y = (2x+1)^5(x^3 - x + 1)^4$.

Example 7. Differentiate $y = e^{\sin x}$.

Theorem 3.4.3. *The derivative of the exponential function is*

$$\frac{d}{dx}(b^x) = b^x \ln b.$$

Proof. Since

$$b^x = (e^{\ln b})^x = e^{(\ln b)x},$$

the Chain Rule gives

$$\begin{aligned}
\frac{d}{dx}(b^x) &= \frac{d}{dx}(e^{(\ln b)x}) \\
&= e^{(\ln b)x}\frac{d}{dx}(\ln b)x \\
&= e^{(\ln b)x} \cdot \ln b \\
&= b^x \ln b.
\end{aligned}$$

□

Example 8. Find $\frac{d}{dx}(2^x)$.

Example 9. Find $f'(x)$ if $f(x) = \sin(\cos(\tan x))$.

Example 10. Differentiate $y = e^{\sec 3\theta}$.

3.5 Implicit Differentiation

Definition 3.5.1. Implicit differentiation is the method of differentiation both sides of an equation with respect to x, and then solving the equation for y' when $y = f(x)$.

Example 1. (a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

Example 2. (a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

(c) At what point in the first quadrant is the tangent line horizontal?

Example 3. Find y' if $\sin(x + y) = y^2 \cos x$.

Example 4. Find y'' if $x^4 + y^4 = 16$.

Theorem 3.5.1. *The derivative of the arcsine function is*

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

Proof. Since $y = \sin^{-1} x$ means $\sin y = x$ and $-\pi/2 \leq y \leq \pi/2$, we have $\cos y \geq 0$. Thus we can differentiate to obtain

$$\begin{aligned} \sin y &= x \\ \cos y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y} \\ &= \frac{1}{\sqrt{1-\sin^2 y}} \\ &= \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

□

Theorem 3.5.2. *The derivative of the arctangent function is*

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

Proof. If $y = \tan^{-1} x$, then $\tan y = x$. Differentiating then gives us

$$\begin{aligned} \tan y &= x \\ \sec^2 y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} \\ &= \frac{1}{1+\tan^2 y} \\ &= \frac{1}{1+x^2}. \end{aligned}$$

□

Example 5. Differentiate

(a) $y = \dfrac{1}{\sin^{-1} x}$

(b) $f(x) = x \arctan \sqrt{x}$.

Theorem 3.5.3. *The derivatives of the Inverse Trigonometric Functions are*

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}.$$

Theorem 3.5.4. *Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Then f^{-1} has derivative*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.

Proof. Since $(f \circ f^{-1})(x) = x$, we have, by the chain rule,

$$\begin{aligned} (f \circ f^{-1})(x) &= x \\ (f \circ f^{-1})'(x) &= 1 \\ f'(f^{-1}(x))(f^{-1})'(x) &= 1 \\ (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))}. \end{aligned}$$

□

Example 6. If $f(4) = 5$ and $f'(4) = \frac{2}{3}$, find $(f^{-1})'(5)$.

3.6 Derivatives of Logarithmic Functions

Theorem 3.6.1. *The derivative of the logarithm function is*

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}.$$

Proof. Let $y = \log_b x$. Then $b^y = x$, so by differentiating we get

$$\begin{aligned} b^y &= x \\ b^y(\ln b)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{b^y \ln b} \\ &= \frac{1}{x \ln b}. \end{aligned}$$

□

Theorem 3.6.2. *The derivative of the natural logarithm is*

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Example 1. Differentiate $y = \ln(x^3 + 1)$.

Example 2. Find $\dfrac{d}{dx} \ln(\sin x)$.

Example 3. Differentiate $f(x) = \sqrt{\ln x}$.

Example 4. Differentiate $f(x) = \log_{10}(2 + \sin x)$.

Example 5. Find $\dfrac{d}{dx} \ln \dfrac{x+1}{\sqrt{x-2}}$.

Example 6. Find $f'(x)$ if $f(x) = \ln |x|$.

Definition 3.6.1. Logarithmic differentiation is the method of calculating derivatives of functions by taking logarithms, differentiating implicitly, and then solving the resulting equation for the derivative.

Example 7. Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

Theorem 3.6.3 (The Power Rule)**.** *If n is any real number and $f(x) = x^n$, then*

$$f'(x) = nx^{n-1}.$$

Proof. Let $y = x^n$. By logarithmic differentiation we get

$$\begin{aligned} y &= x^n \\ \ln|y| &= \ln|x|^n \\ &= n\ln|x| \qquad x \neq 0 \\ \frac{y'}{y} &= \frac{n}{x} \\ y' &= n\frac{y}{x} \\ &= n\frac{x^n}{x} \\ &= nx^{n-1}. \end{aligned}$$

□

Example 8. Differentiate $y = x^{\sqrt{x}}$.

Theorem 3.6.4. *The number e can be defined as the limit*

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Proof. If $f(x) = \ln x$, then $f'(1) = 1$, so

$$\begin{aligned} f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{x\to 0}\frac{f(1+x)-f(1)}{x} \\ &= \lim_{x\to 0}\frac{\ln(1+x)-\ln 1}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x) \\ &= \lim_{x\to 0}\ln(1+x)^{1/x} = 1. \end{aligned}$$

Thus

$$e = e^1 = e^{\left(\lim\limits_{x\to 0}\ln(1+x)^{1/x}\right)} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}} = \lim_{x\to 0}(1+x)^{1/x}.$$

Then if we let $n = 1/x$, $n \to \infty$ as $x \to 0^+$, so we are done. □

3.7 Rates of Change in the Sciences

Example 1. The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters.

(a) Find the velocity at time t.

(b) What is the velocity after 2 s? After 4 s?

(c) When is the particle at rest?

(d) When is the particle moving forward (that is, in the positive direction)?

(e) Draw a diagram to represent the motion of the particle.

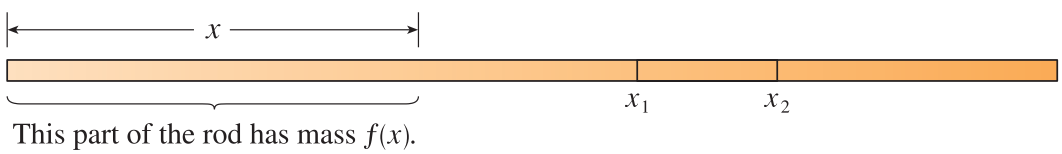
(f) Find the total distance traveled by the particle during the first five seconds.

(g) Find the acceleration at time t and after 4 s.

(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$.

(i) When is the particle speeding up? When is it slowing down?

Example 2. If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length ($\rho = m/l$) and measured in kilograms per meter. Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point x is $m = f(x)$, as shown in the figure.



In this case the average density is the average rate of change over a given interval, and the linear density is the limit of these average densities.
If $m = f(x) = \sqrt{x}$, where x is measured in meters and m in kilograms, find the average density of the part of the rod given by $1 \leq x \leq 1.2$ and the density at $x = 1$.

Example 3. The average current during a time interval is the average rate of change of the net charge over that interval, and the current at a given time is the limit of the average current (the rate at which charge flows through a surface, measured in units of charge per unit time). The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds) is given by $Q(t) = t^3 - 2t^2 + 6t + 2$. [The unit of current is an ampere (1 A = 1 C/s).] Find the current when

(a) $t = 0.5$ s

(b) $t = 1$ s.

At what time is the current lowest?

Example 4. The concentration of a reactant A is the number of moles (1 mole = 6.022×10^{23} molecules) per liter and is denoted by [A] for a chemical reaction

$$\mathrm{A} + \mathrm{B} \to \mathrm{C}.$$

The average rate of reaction during a time interval is the average rate of change of the concentration of the product [C] over that interval, and the rate of reaction at a given time is the limit of the average rate of reaction.
If one molecule of a product C is formed from one molecule of a reactant A and one molecule of a reactant B, and the initial concentrations of A and B have a common value [A] = [B] = a moles/L, then

$$[\mathrm{C}] = \frac{a^2kt}{akt+1}$$

where k is a constant.

(a) Find the rate of reaction at time t.

(b) Show that if $x = [\mathrm{C}]$, then

$$\frac{dx}{dt} = k(a-x)^2.$$

(c) What happens to the concentration as $t \to \infty$?

(d) What happens to the rate of reaction as $t \to \infty$?

(e) What do the results of parts (c) and (d) mean in practical terms?

Example 5. If a given substance is kept a constant temperature, then the rate of change of its volume V with respect to its pressure P is the derivative dV/dP. The compressibility is defined by

$$\text{isothermal compressibility} = \beta = -\frac{1}{V}\frac{dV}{dP}.$$

The volume V (in cubic meters) of a sample of air at 25°C was found to be related to the pressure P (in kilopascals) by the equation

$$V = \frac{5.3}{P}.$$

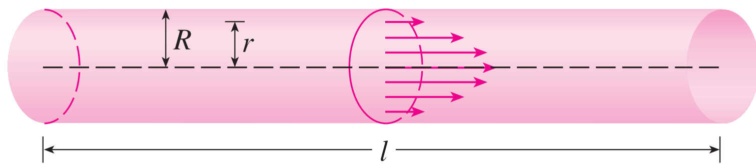
Determine the compressibility when $P = 50$ kPa.

Example 6. Let $n = f(t)$ be the number of individuals in an animal or plant population at time t. The average rate of growth during a time period is the average rate of change of the growth of the population over that time period, and the rate of growth at a given time is the limit of the average rate of growth.
Suppose that a population of bacteria doubles every hour. The population function representing the bacteria's growth can be found to be

$$n = n_0 2^t$$

where n_0 is the initial population and the time t is measured in hours.
Find the rate of growth for a colony of bacteria with an initial population $n_0 = 100$ after 4 hours.

Example 7. The shape of a blood vessel can be modeled by a cylindrical tube with radius R and length l as illustrated in the figure.



The relationship between the velocity v of the blood and the distance r from the axis is given by the law of laminar flow

$$v = \frac{P}{4\eta l}(R^2 - r^2)$$

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P and l are constant, then v is a function of r with domain $[0, R]$. The velocity gradient at a given time is the limit of the average rate of change of the velocity.
For one of the smaller human arteries we can take $\eta = 0.027$, $R = 0.008$ cm, $l = 2$ cm, and $P = 4000$ dynes/cm^2. Find the speed at which blood is flowing at $r = 0.002$ and find the velocity gradient at that point.

Example 8. Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C is called a cost function. The instantaneous rate of change of cost with respect to the number of items produced, called the marginal cost, is the limit of the average rate of change of the cost.
Suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10,000 + 5x + 0.01x^2.$$

Find the marginal cost at the production level of 500 items and compare it to the actual cost of producing the 501st item.

3.8 Exponential Growth and Decay

Definition 3.8.1. The equation

$$\frac{dy}{dt} = ky$$

is called the law of natural growth (if $k > 0$) or the law of natural decay (if $k < 0$). It is called a differential equation because it involves an unknown function y and its derivative dy/dt.

Theorem 3.8.1. *The only solutions of the differential equation $dy/dt = ky$ are the exponential functions*

$$y(t) = y(0)e^{kt}.$$

Definition 3.8.2. If $P(t)$ is the size of a population at time t, then

$$k = \frac{1}{P}\frac{dP}{dt}$$

is the growth rate divided by population, called the relative growth rate.

Example 1. Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

Definition 3.8.3. If $m(t)$ is the mass remaining from an initial mass m_0 of a substance after time t, then the relative decay rate is

$$-\frac{1}{m}\frac{dm}{dt}.$$

It follows that the mass decays exponentially according to the equation

$$m(t) = m_0 e^{kt},$$

where the rate of decay is expressed in terms of half-life, the time required for half of any given quantity to decay.

Example 2. The half-life of radium-226 is 1590 years.

(a) A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of the sample that remains after t years.

(b) Find the mass after 1000 years correct to the nearest milligram.

(c) When will the mass be reduced to 30 mg?

Example 3. Newton's Law of Cooling can be represented as a differential equation

$$\frac{dT}{dt} = k(T - T_s),$$

where T is the temperature of the object at time t and T_s is the temperature of the surroundings. The exponential function $y(t) = y(0)e^{kt}$ is a solution to this differential equation when $y(t) = T(t) - T_s$.
A bottle of soda pop at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F. After half an hour the soda pop has cooled to 61°F.

(a) What is the temperature of the soda pop after another half hour?

(b) How long does it take for the soda pop to cool to 50°F?

Example 4. In general, if an amount A_0 is invested at an interest rate r, then after t years it is worth $A_0(1+r)^t$. Usually, however, interest is compounded more frequently, say, n times a year. Then in each compounding period the interest rate is r/n and there are nt compounding periods in t years, so the value of the investment is

$$A_0\left(1+\frac{r}{n}\right)^{nt}.$$

Therefore, taking limits gives us the amount after t years as

$$A(t) = A_0 e^{rt}$$

when interest is continuously compounded. Determine the value of an investment of \$1000 after 3 years of continuously compounding 6% interest. Compare this to the value of the same investment compounded annually instead.

3.9 Related Rates

Example 1. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Example 2. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Example 3. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

Example 4. Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

Example 5. A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

3.10 Linear Approximations and Differentials

Definition 3.10.1. The approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the linear approximation or tangent line approximation of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization of f at a.

Example 1. Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

Example 2. For what values of x is the linear approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

accurate to within 0.5? What about accuracy to within 0.1?

Definition 3.10.2. If $y = f(x)$, where f is a differentiable function, then the differential dx is an independent variable; that is, dx can be given the value of any real number. The differential dy is then defined in terms of dx by the equation

$$dy = f'(x)dx.$$

Example 3. Compare the values Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes

(a) from 2 to 2.05

(b) from 2 to 2.01.

Example 4. The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

3.11 Hyperbolic Functions

Definition 3.11.1. Functions that have the same relationship to the hyperbola that trigonometric functions have to the circle are called hyperbolic functions and are defined as follows

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}.$$

Theorem 3.11.1 (Hyperbolic Identities)**.**

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$
$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$$

Example 1. Prove

(a) $\cosh^2 x - \sinh^2 x = 1$

(b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

Theorem 3.11.2 (Derivatives of Hyperbolic Functions)**.**

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$
$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x.$$

Example 2. Find $\dfrac{d}{dx}(\cosh \sqrt{x})$.

Theorem 3.11.3 (Inverse Hyperbolic Functions)**.**

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}) \qquad x \in \mathbb{R}$$
$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1}) \qquad x \geq 1$$
$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -1 < x < 1.$$

Example 3. Show that $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$.

Theorem 3.11.4 (Derivatives of Inverse Hyperbolic Functions)**.**

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}}$$
$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}.$$

Example 4. Prove that $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$.

Example 5. Find $\frac{d}{dx}[\tanh^{-1}(\sin x)]$.

Chapter 4

Applications of Differentiation

4.1 Maximum and Minimum Values

Definition 4.1.1. Let c be a number in the domain D of a function f. Then $f(c)$ is the absolute maximum value (or global maximum value) of f on D if $f(c) \geq f(x)$ for all x in D and $f(c)$ is the absolute minimum value (or global minimum value) of f on D if $f(c) \leq f(x)$ for all x in D. These values are called extreme values of f.

Definition 4.1.2. The number $f(c)$ is a local maximum value of f if $f(c) \geq f(x)$ when x is near c and a local minimum value of f if $f(c) \leq f(x)$ when x is near c. When we say near, we mean on an open interval containing c. These values are called local extreme values of f.

Example 1. For what values of x does $f(x) = \cos x$ take on its maximum and minimum values?

Example 2. Find all of the extreme values of $f(x) = x^2$.

Example 3. Find all of the extreme values of $f(x) = x^3$.

Example 4. Find all of the extreme values of $f(x) = 3x^4 - 16x^3 + 18x^2$ within the domain $-1 \leq x \leq 4$.

Theorem 4.1.1 (Extreme Value Theorem)**.** *If f is continuous on a closed interval $[a, b]$ then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.*

Theorem 4.1.2 (Fermat's Theorem)**.** *If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.*

Proof. Suppose f has a local maximum at c. Then, by definition, $f(c) \geq f(x)$ if x is near c, so if we let $h > 0$ be close to 0 we have

$$\begin{aligned} f(c) &\geq f(c+h) \\ f(c+h) - f(c) &\leq 0 \\ \frac{f(c+h) - f(c)}{h} &\leq \frac{0}{h} \\ \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} &\leq \lim_{h \to 0^+} 0 \\ f'(c) &\leq 0 \end{aligned}$$

If $h < 0$ the direction of the inequality is reversed and we get $f'(c) \geq 0$. Thus combining these inequalities gives us $f'(c) = 0$. A similar argument can be used to achieve the same result if f has a local minimum at c. □

Example 5. Use the function $f(x) = x^3$ to determine whether the converse of Fermat's theorem is true.

Example 6. Does Fermat's theorem apply to the function $f(x) = |x|$?

Definition 4.1.3. A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 7. Find the critical numbers of $x^{3/5}(4 - x)$.

Example 8. Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \leq x \leq 4.$$

Example 9. (a) Use a graphing device to estimate the absolute minimum and maximum values of the function $f(x) = x - 2\sin x$, $0 \leq x \leq 2\pi$.

(b) Use calculus to find the exact minimum and maximum values.

Example 10. The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery*. A model for the velocity of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ seconds, is given by

$$v(t) = 0.001302t^3 - 0.09029t^2 + 23.61t - 3.083$$

(in feet per second). Using this model, estimate the absolute maximum and minimum values of the acceleration of the shuttle between liftoff and the jettisoning of the boosters.

4.2 The Mean Value Theorem

Theorem 4.2.1 (Rolle's Theorem). *Let f be a function that satisfies the following three hypotheses:*

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*
3. *$f(a) = f(b)$.*

Then there is a number c in (a, b) such that $f'(c) = 0$.

Proof. If $f(x) = k$, a constant, then $f'(x) = 0$ for all $x \in (a, b)$. If $f(x) > f(a)$ for some $x \in (a, b)$ then f has a local maximum for a number $c \in (a, b)$ by the extreme value theorem. Since f is differentiable on (a, b), $f'(c) = 0$ by Fermat's theorem. By the same reasoning, $f'(c) = 0$ if $f(x) < f(a)$. □

Example 1. How could Rolle's theorem be applied to a position function that models a ball thrown upward?

Example 2. Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Theorem 4.2.2 (The Mean Value Theorem)**.** *Let f be a function that satisfies the following hypotheses:*

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Let h be the difference between f and the secant line to f on $[a, b]$, i.e.,

$$h(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right].$$

Then h is continuous on $[a, b]$ and differentiable on (a, b) because it is the sum of f and a first-degree polynomial, which are both continuous on $[a, b]$ and differentiable on (a, b). Also,

$$h(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0$$
$$h(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = 0,$$

so $h(a) = h(b)$. Therefore, by Rolle's thereom, there is a number c in (a, b) such that $h'(c) = 0$, i.e.,

$$h'(c) = 0 = f'(c) - \frac{f(b) - f(a)}{b - a},$$

which is equivalent to

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

as desired. □

Example 3. Find a number c in $(0, 2)$ such that the slope of the secant line is equal to the slope of the tangent line for the function $f(x) = x^3 - x$.

Example 4. What does the mean value theorem say about the velocity of an object moving in a straight line?

Example 5. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

Theorem 4.2.3. *If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).*

Proof. Let $x_1, x_2 \in (a, b)$ such that $x_1 \neq x_2$. By the mean value theorem for f on $[x_1, x_2]$ we get

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1),$$

for some $c \in (a, b)$. But $f'(x) = 0$ for all x in this interval, so $f(x_2) = f(x_1)$ for all x in this interval. Since x_1 and x_2 were chosen arbitrarily, f is constant on (a, b). □

Corollary 4.2.1. *If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is $f(x) = g(x) + c$ where c is a constant.*

Proof. Let

$$F(x) = f(x) - g(x).$$

Then

$$F'(x) = f'(x) - g'(x) = 0,$$

so F is constant by the previous theorem, and thus $f - g$ is constant. □

Example 6. Prove the identity $\tan^{-1} x + \cot^{-1} x = \pi/2$.

4.3 Derivatives and the Shape of a Graph

Theorem 4.3.1 (Increasing/Decreasing Test)**.**

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Proof. Let x_1, x_2 be two numbers on an interval where $f'(x) > 0$ such that $x_1 < x_2$. Then by the mean value theorem,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c in the interval. But $f'(c) > 0$ and $x_2 - x_1 > 0$, so $f(x_2) - f(x_1) > 0$, i.e.,

$$f(x_2) > f(x_1)$$

in the interval. Since x_1 and x_2 were chosen arbitrarily, we are done, and the second half of the theorem is proved similarly. □

Example 1. Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Theorem 4.3.2 (The First Derivative Test)**.** *Suppose that c is a critical number of a continuous function f.*

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and to the right of c, or negative to the left and to the right of c, then f has no local minimum or maximum at c.

Example 2. Find the local minimum and maximum values of the function f in Example 1.

Example 3. Find the local maximum and minimum values of the function

$$g(x) = x + 2\sin x \qquad 0 \leq x \leq 2\pi.$$

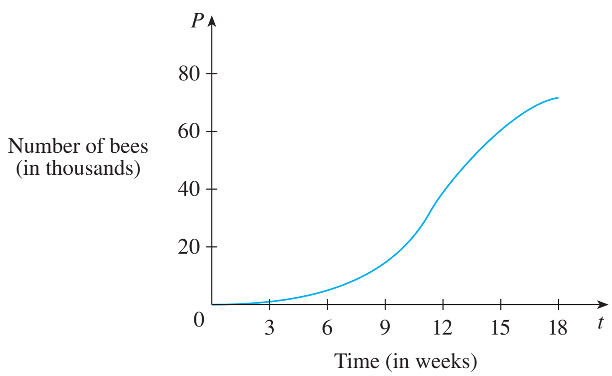
Definition 4.3.1. If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I. If the graph of f lies below all of its tangents on I, it is called concave downward on I.

Theorem 4.3.3 (Concavity Test)**.**

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Example 4. The figure shows a population graph for Cyprian honeybees raised in an apiary. How does the rate of population increase change over time? When is this rate highest? Over what intervals is P concave upward or concave downward?



Definition 4.3.2. A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Example 5. Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$.

(ii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$.

(iii) $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$.

Theorem 4.3.4 (The Second Derivative Test)**.** *Suppose f' is continuous near c.*

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example 6. Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Example 7. Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$.

Example 8. Use the first and second derivatives of $f(x) = e^{1/x}$, together with asymptotes, to sketch its graph.

4.4 Indeterminate Forms and l'Hospital's Rule

Theorem 4.4.1 (L'Hospital's Rule)**.** *Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that*

$$\lim_{x\to a} f(x) = 0 \qquad \textit{and} \qquad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \qquad \textit{and} \qquad \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Example 1. Find $\displaystyle\lim_{x\to 1} \frac{\ln x}{x-1}$.

Example 2. Calculate $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2}$.

Example 3. Calculate $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$.

Example 4. Find $\lim_{x\to 0} \dfrac{\tan x - x}{x^3}$.

Example 5. Find $\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x}$.

Example 6. Evaluate $\lim_{x\to 0^+} x\ln x$.

Example 7. Compute $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$.

Example 8. Calculate $\lim_{x\to\infty}(e^x - x)$.

Example 9. Calculate $\lim_{x\to 0^+} (1 + \sin 4x)^{\cot x}$.

Example 10. Find $\lim_{x\to 0^+} x^x$.

4.5 Summary of Curve Sketching

Use the following guidelines when sketching curves by hand:

A. Domain

B. Intercepts

C. Symmetry

D. Asymptotes

E. Intervals of Increase or Decrease

F. Local Maximum and Minimum Values

G. Concavity and Points of Inflection

Example 1. Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

Example 2. Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x^2+1}}$.

Example 3. Sketch the graph of $f(x) = xe^x$.

Example 4. Sketch the graph of $f(x) = \dfrac{\cos x}{2 + \sin x}$.

Example 5. Sketch the graph of $y = \ln(4 - x^2)$.

Definition 4.5.1. If

$$\lim_{x\to\infty}[f(x) - (mx + b)] = 0$$

where $m \neq 0$, then the line $y = mx + b$ is called a slant asymptote because the vertical distance between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0.

Example 6. Sketch the graph of $f(x) = \dfrac{x^3}{x^2 + 1}$.

4.6 Graphing with Calculus and Calculators

Example 1. Graph the polynomial $f(x) = 2x^6 + 3x^5 + 3x^3 - 2x^2$. Use the graphs of f' and f'' to estimate all maximum and minimum points and intervals of concavity.

Example 2. Draw the graph of the function

$$f(x) = \frac{x^2 + 7x + 3}{x^2}$$

in a viewing rectangle that contains all the important features of the function. Estimate the maximum and minimum values and the intervals of concavity. Then use calculus to find these quantities exactly.

Example 3. Graph the function $f(x) = \dfrac{x^2(x+1)^3}{(x-2)^2(x-4)^4}$.

Example 4. Graph the function $f(x) = \sin(x + \sin 2x)$. For $0 \leq x \leq \pi$, estimate all maximum and minimum values, intervals of increase and decrease, and inflection points.

Example 5. How does the graph of $f(x) = 1/(x^2 + 2x + c)$ vary as c varies?

4.7 Optimization Problems

Example 1. A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Example 2. A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

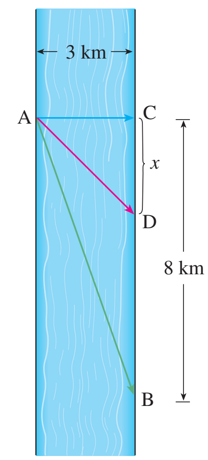
Theorem 4.7.1 (First Derivative Test for Absolute Extreme Values)**.** *Suppose that c is a critical number of a continuous function f defined on an interval.*

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Example 3. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

Example 4. A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see the figure). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)



Example 5. Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Definition 4.7.1. If $p(x)$ is the price per unit that a company can charge if it sells x units, then p is called the demand function (or price function).
If x units are sold, then the total revenue

$$R(x) = \text{quantity} \times \text{price} = xp(x)$$

and R is called the revenue function. The derivative R' of the revenue function is called the marginal revenue function and is the rate of change of revenue with respect to the number of units sold.
If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

where C is the cost function and P is called the profit function. The marginal profit function is P', the derivative of the profit function.

Example 6. A store has been selling 200 flat-screen TVs a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of TVs sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

4.8 Newton's Method

Theorem 4.8.1 (Newton's Method)**.** *If x_n is the nth approximation of a root r for a function f then*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example 1. Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

Example 2. Use Newton's method to find $\sqrt[6]{2}$ to eight decimal places.

Example 3. Find, correct to six decimal places, the root of the equation $\cos x = x$.

4.9 Antiderivatives

Definition 4.9.1. A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem 4.9.1. *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is*

$$F(x) + C$$

where C is an arbitrary constant.

Proof. Follows by Corollary 4.2.1 to the mean value theorem. □

Example 1. Find the most general antiderivative of each of the following functions.

(a) $f(x) = \sin x$

(b) $f(x) = 1/x$

(c) $f(x) = x^n$, $n \neq -1$

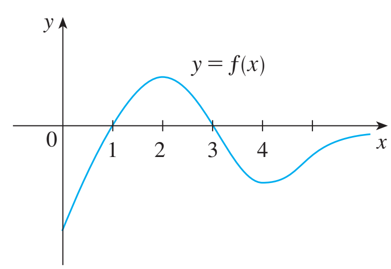
Example 2. Find all functions g such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}.$$

Example 3. Find f if $f'(x) = e^x + 20(1 + x^2)^{-1}$ and $f(0) = -2$.

Example 4. Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

Example 5. The graph of a function f is given in the figure. Make a rough sketch of an antiderivative F, given that $F(0) = 2$.



Example 6. A particle moves in a straight line and has acceleration given by $a(t) = 6t+4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

Example 7. A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground? [For motion close to the ground we may assume that the downward acceleration g is constant, its value being about 9.8 m/s^2 (or 32 ft/s^2).]

Chapter 5

Integrals

5.1 Areas and Distances

Example 1. Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

Example 2. For the region in Example 1, show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} R_n = \frac{1}{3}.$$

Definition 5.1.1. The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

The last equality is an example of the use of sigma notation to write sums with many terms more compactly.

Definition 5.1.2. Numbers x_i^* in the ith subinterval $[x_{i-1}, x_i]$ are called sample points. We form lower (and upper) sums by choosing the sample points x_i^* so that $f(x_i^*)$ is the minimum (and maximum) value of f on the ith subinterval.

Example 3. Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

(b) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

Example 4. Suppose the odometer on a car is broken. Estimate the distance driven in feet over a 30-second time interval by using the speedometer readings taken every five seconds and recorded in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

5.2 The Definite Integral

Definition 5.2.1. If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0(= a), x_1, x_2, \ldots, x_n(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is integrable on $[a, b]$.

Definition 5.2.2. The symbol $\int$ is called an integral sign. In the notation $\int_a^b f(x)dx$, $f(x)$ is called the integrand and a and b are called the limits of integration; a is the lower limit and b is the upper limit. The procedure of calculating an integral is called integration.

Definition 5.2.3. The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

is called a Riemann sum and it can be used to approximate the definite integral of an integrable function within any desired degree of accuracy.

Definition 5.2.4. A definite integral can be interpreted as a net area, that is, a difference of areas:

$$\int_a^b f(x)dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and the above the graph of f.

Theorem 5.2.1. *If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.*

Theorem 5.2.2. *If f is integrable on $[a, b]$, then*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n} \qquad \textit{and} \qquad x_i = a + i\Delta x.$$

Example 1. Express

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x$$

as an integral on the interval $[0, \pi]$.

Theorem 5.2.3. *The following formulas are true when working with sigma notation:*

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i.$$

Example 2. (a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

(b) Evaluate $\int_0^3 (x^3 - 6x)dx$.

Example 3. (a) Set up an expression for $\int_1^3 e^x dx$ as a limit of sums.

(b) Use a computer algebra system to evaluate the expression.

Example 4. Evaluate the following integrals by interpreting each in terms of areas.

(a) $\int_0^1 \sqrt{1-x^2}dx$

(b) $\int_0^3 (x-1)dx$

Theorem 5.2.4 (Midpoint Rule)**.**

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x[f(\bar{x}_1) + \cdots + f(\bar{x}_n)]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \textit{midpoint of } [x_{i-1}, x_i].$$

Example 5. Use the Midpoint Rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x}dx$.

Theorem 5.2.5 (Properties of the Definite Integral)**.**

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx.$
2. $\int_a^a f(x)dx = 0.$
3. $\int_a^b c dx = c(b-a)$, *where c is any constant.*
4. $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$
5. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, *where c is any constant.*
6. $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$
7. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx.$

Example 6. Use the properties of integrals to evaluate $\int_0^1 (4+3x^2)dx$.

Example 7. If it is known that $\int_0^{10} f(x)dx = 17$ and $\int_0^{8} f(x)dx = 12$, find $\int_8^{10} f(x)dx$.

Theorem 5.2.6 (Comparison Properties of the Integral)**.**

8. *If* $f(x) \geq 0$ *for* $a \leq x \leq b$*, then* $\int_a^b f(x)dx \geq 0$*.*

9. *If* $f(x) \geq g(x)$ *for* $a \leq x \leq b$*, then* $\int_a^b f(x)dx \geq \int_a^b g(x)dx$*.*

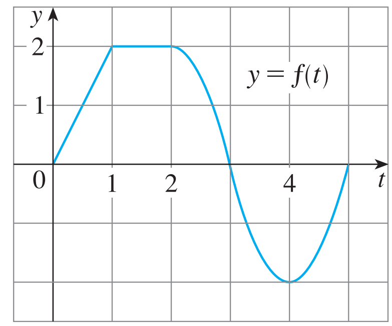
10. *If* $m \leq f(x) \leq M$ *for* $a \leq x \leq b$*, then*

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Example 8. Use Property 10 to estimate $\int_0^1 e^{-x^2}dx$.

5.3 The Fundamental Theorem of Calculus

Example 1. If f is the function whose graph is shown in the figure and $g(x) = \int_0^x f(t)dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$. Then sketch a rough graph of g.



Theorem 5.3.1 (The Fundamental Theorem of Calculus, Part 1)**.** *If f is continuous on $[a, b]$, then the function g defined by*

$$g(x) = \int_a^x f(t)dt \qquad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Example 2. Find the derivative of the function $g(x) = \displaystyle\int_0^x \sqrt{1 + t^2}dt$.

Example 3. Find the derivative of the Fresnel function

$$S(x) = \int_0^x \sin(\pi t^2/2)dt$$

and compare its graph with that of $S(x)$ to visually confirm the fundamental theorem of calculus.

Example 4. Find $\frac{d}{dx}\int_1^{x^4} \sec t dt$.

Theorem 5.3.2 (The Fundamental Theorem of Calculus, Part 2)**.** *If f is continuous on $[a, b]$, then*

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Proof. By Part 1, $g'(x) = f(x)$; that is, g is an antiderivative of f. If F is any other antiderivative of f on $[a, b]$, then, by Corollary 4.2.1,

$$F(x) = g(x) + C$$

for $a < x < b$. By continuity, this is also true for $x \in [a, b]$, so again by Part 1,

$$g(a) = \int_a^a f(t)dt = 0$$

and thus

$$\begin{aligned} F(b) - F(a) &= [g(b) + C] - [g(a) + C] \\ &= g(b) + C - 0 - C \\ &= g(b) \\ &= \int_a^b f(t)dt. \end{aligned}$$ □

Example 5. Evaluate the integral $\int_1^3 e^x dx$.

Example 6. Find the area under the parabola $y = x^2$ from 0 to 1.

Example 7. Evaluate $\displaystyle\int_3^6 \frac{dx}{x}$.

Example 8. Find the area under the cosine curve from 0 to b, where $0 \leq b \leq \pi/2$.

Example 9. What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2} dx = \left. \frac{x^{-1}}{-1} \right]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

5.4 Indefinite Integrals and Net Change Theorem

Definition 5.4.1. An antiderivative of f is called an indefinite integral where

$$\int f(x)dx = F(x) \qquad \text{means} \qquad F'(x) = f(x).$$

Example 1. Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)dx.$$

Example 2. Evaluate $\int \frac{\cos\theta}{\sin^2\theta}d\theta$.

Example 3. Evaluate $\int_0^3 (x^3 - 6x)dx$.

Example 4. Find $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx$ and interpret the result in terms of areas.

Example 5. Evaluate $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2} dt$.

Theorem 5.4.1 (Net Change Theorem)**.** *The integral of a rate of change is the net change:*

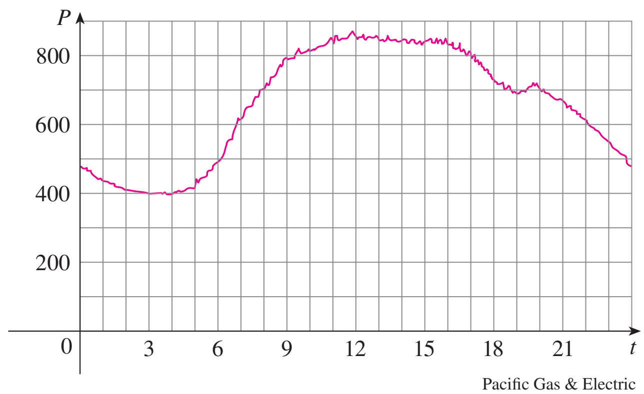
$$\int_a^b F'(x)dx = F(b) - F(a).$$

Example 6. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

(b) Find the distance traveled during this time period.

Example 7. The figure shows the power consumption in the city of San Francisco for a day in September (P is measured in megawatts; t is measured in hours starting at midnight). Estimate the energy used on that day.



5.5 The Substitution Rule

Theorem 5.5.1 (The Substitution Rule)**.** *If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then*

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Proof. If $f = F'$, then, by the Chain Rule,

$$\frac{d}{dx}[F(g(x))] = f(g(x))g'(x).$$

Thus if $u = g(x)$, then we have

$$\int f(g(x))g'(x)dx = F(g(x)) + C = F(u) + C = \int f(u)du.$$

□

Example 1. Find $\int x^3 \cos(x^4 + 2)dx$.

Example 2. Evaluate $\int \sqrt{2x+1}dx$.

Example 3. Find $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}dx$.

Example 4. Calculate $\displaystyle\int e^{5x}dx$.

Example 5. Find $\displaystyle\int \sqrt{1+x^2}x^5dx$.

Example 6. Calculate $\int \tan x dx$.

Theorem 5.5.2 (The Substitution Rule for Definite Integrals)**.** *If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Proof. Let F be an antiderivative of f. Then $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$, so by part 2 of the fundamental theorem of calculus, we have

$$\int_a^b f(g(x))g'(x)dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a)).$$

By applying part 2 a second time, we also have

$$\int_{g(a)}^{g(b)} f(u)du = F(u)\Big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a)).$$ □

Example 7. Evaluate $\int_0^4 \sqrt{2x+1}dx$.

Example 8. Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$.

Example 9. Calculate $\int_1^e \frac{\ln x}{x}dx$.

Theorem 5.5.3 (Integrals of Symmetric Functions)**.** *Suppose f is continuous on $[-a, a]$.*

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)dx = 0.$

Proof. First we split the integral:

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = -\int_{0}^{-a} f(x)dx + \int_{0}^{a} f(x)dx.$$

By substituting $u = -x$ we get $du = -dx$ and $u = a$ when $x = -a$, so

$$-\int_{0}^{-a} f(x)dx = -\int_{0}^{a} f(-u)(-du) = \int_{0}^{a} f(-u)du$$

and therefore

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(-u)du + \int_{0}^{a} f(x)dx.$$

(a) If f is even then $f(-u) = f(u)$, so

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(u)du + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

(b) If f is odd then $f(-u) = -f(u)$, so

$$\int_{-a}^{a} f(x)dx = -\int_{0}^{a} f(u)du + \int_{0}^{a} f(x)dx = 0.$$

□

Example 10. Evaluate $\int_{-2}^{2}(x^6+1)dx$.

Example 11. Evaluate $\int_{-1}^{1}\frac{\tan x}{1+x^2+x^4}dx$.

Chapter 6

Applications of Integration

6.1 Areas Between Curves

Definition 6.1.1. The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

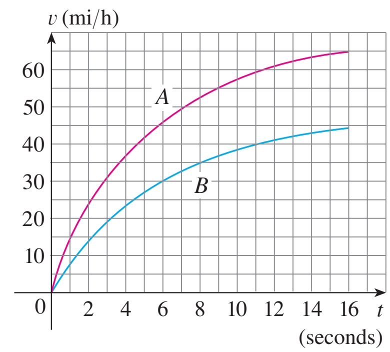
$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x = \int_a^b [f(x) - g(x)]dx.$$

Example 1. Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

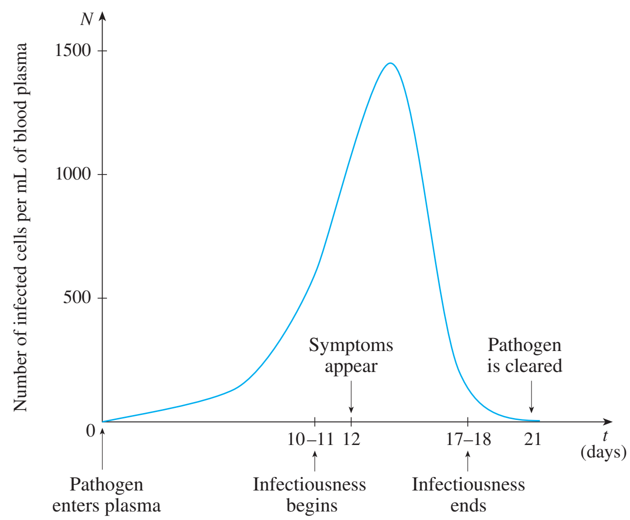
Example 2. Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Example 3. Find the approximate area of the region bounded by the curves $y = x/\sqrt{x^2+1}$ and $y = x^4 - x$.

Example 4. The figure shows the velocity curves for two cars, A and B, that start side by side and move along the same road. What does the area between the curves represent? Use the Midpoint Rule to estimate it.



Example 5. The figure is an example of a pathogenesis curve for a measles infection. It shows how the disease develops in an individual with no immunity after the measles virus spreads to the bloodstream from the respiratory tract.



The patient becomes infectious to others once the concentration of infected cells becomes great enough, and he or she remains infectious until the immune system manages to prevent further transmission. However, symptoms don't develop until the "amount of infection" reaches a particular threshold. The amount of infection needed to develop symptoms depends on both the concentration of infected cells and time, and corresponds to the area under the pathogenesis curve until symptoms appear.

(a) The pathogenesis curve in the figure has been modeled by $f(t) = -t(t-21)(t+1)$. If infectiousness begins on day $t_1 = 10$ and ends on day $t_2 = 18$, what are the corresponding concentration levels of infected cells?

(b) The level of infectiousness for an infected person is the area between $N = f(t)$ and the line through the points $P_1(t_1, (f(t_1))$ and $P_2(t_2, f(t_2))$, measured in (cells/mL)$\cdot$ days. Compute the level of infectiousness for this particular patient.

Definition 6.1.2. The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)| dx.$$

Example 6. Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

Example 7. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

6.2 Volumes

Definition 6.2.1 (Definition of Volume)**.** Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx.$$

Example 1. Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Example 2. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.

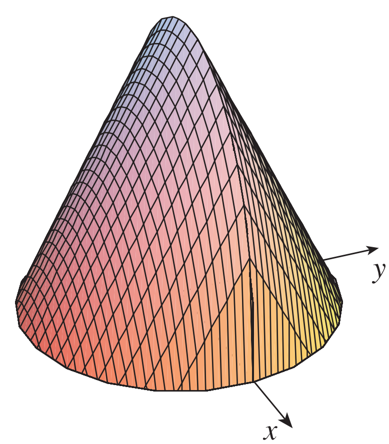
Example 3. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

Example 4. The region $\mathscr{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

Example 5. Find the volume of the solid obtained by rotating the region in Example 4 about the line $y = 2$.

Example 6. Find the volume of the solid obtained by rotating the region in Example 4 about the line $x = -1$.

Example 7. The figure shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.



Example 8. Find the volume of a pyramid whose base is a square with side L and whose height is h.

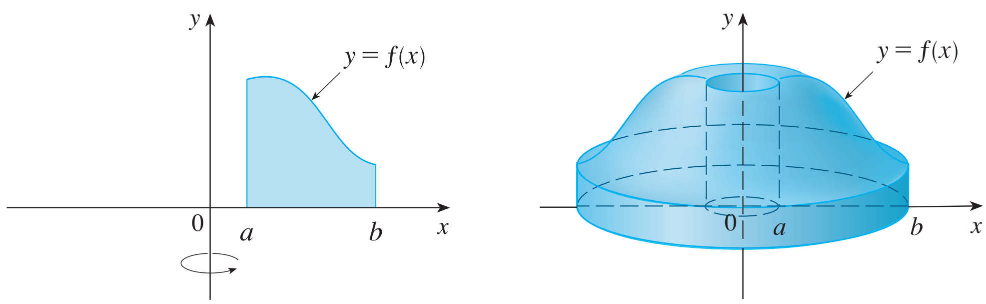
Example 9. A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.

6.3 Volumes by Cylindrical Shells

Theorem 6.3.1 (Method of Cylindrical Shells)**.** *The volume of the solid in the figure, obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is*

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\Delta x = \int_a^b 2\pi x f(x) dx \qquad \textit{where } 0 \le a \le b$$

and where $\bar{x}_i$ is the midpoint of the ith subinterval $[x_{i-1}, x_i]$.



Example 1. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Example 2. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

Example 3. Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

Example 4. Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

6.4 Work

Definition 6.4.1. In general, if an object moves along a straight line with position function $s(t)$, then the force F on the object (in the same direction) is given by Newton's Second Law of Motion as the product of its mass m and its acceleration a:

$$F = ma = m\frac{d^2s}{dt^2}.$$

Definition 6.4.2. In the case of constant acceleration, the force F is also constant and the work done is defined to be the product of the force F and distance d that the object moves:

$$W = Fd \qquad \text{work} = \text{force} \times \text{distance}.$$

Example 1. (a) How much work is done in lifting a 1.2-kg book off the floor to put it on a desk that is 0.7 m high? Use the fact that the acceleration due to gravity is $g = 9.8\text{ m/s}^2$.

(b) How much work is done in lifting a 20-lb weight 6 ft off the ground?

Definition 6.4.3. If the force $f(x)$ on an object is variable, then we define the work done in moving the object from a to b as

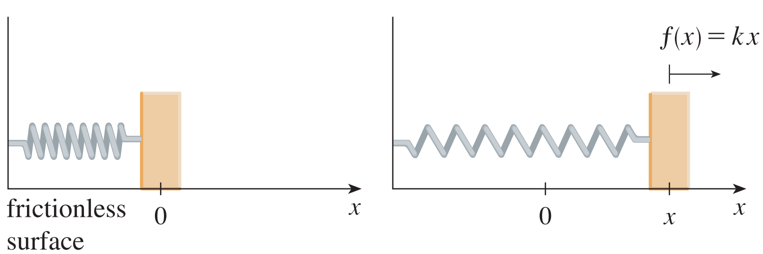
$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx.$$

Example 2. When a particle is located a distance x feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

Theorem 6.4.1 (Hooke's Law)**.** *The force required to maintain a spring stretched x units beyond its natural length is proportional to x:*

$$f(x) = kx$$

where k is a positive constant called the <u>*spring constant*</u> *(see the figure). Hooke's Law holds provided that x is not too large.*



(a) Natural position of spring (b) Stretched position of spring

Example 3. A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

Example 4. A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

Example 5. A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3.)

6.5 Average Value of a Function

Definition 6.5.1. The average value of a function f on the interval $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example 1. Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

Theorem 6.5.1 (The Mean Value Theorem for Integrals)**.** *If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that*

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)dx,$$

that is,

$$\int_a^b f(x)dx = f(c)(b-a).$$

Proof. By applying the Mean Value Theorem for derivatives to the function $F(x) = \int_a^x f(t)dt$, we see that there exists a number c in $[a, b]$ such that

$$\begin{aligned} F'(c) &= \frac{F(b)-F(a)}{b-a} \\ \frac{d}{dx}\left[\int_a^x f(t)dt\right]\Bigg|_c &= \frac{F(b)-F(a)}{b-a} \\ f(c) &= \frac{1}{b-a}[F(b)-F(a)] \\ &= \frac{1}{b-a}\int_a^b f(x)dx. \end{aligned}$$

□

Example 2. Find a number c in the interval $[-1, 2]$ that satisfies the mean value theorem for integrals for the function $f(x) = 1 + x^2$.

Example 3. Show that the average velocity of a car over a time interval $[t_1, t_2]$ is the same as the average of its velocities during the trip.

Chapter 7

Techniques of Integration

7.1 Integration by Parts

Theorem 7.1.1 (Formula for Integration by Parts)**.** *If f and g are differentiable functions then*

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx,$$

or, equivalently,

$$\int udv = uv - \int vdu$$

where $u = f(x)$ and $v = g(x)$.

Proof. By the Product Rule,

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= f(x)g'(x) + g(x)f'(x) \\
f(x)g(x) &= \int [f(x)g'(x) + g(x)f'(x)]dx \\
&= \int f(x)g'(x)dx + \int g(x)f'(x)dx \\
\int f(x)g'(x)dx &= f(x)g(x) - \int g(x)f'(x)dx
\end{aligned}$$

□

Example 1. Find $\int x \sin x dx$.

Example 2. Evaluate $\int \ln x dx$.

Example 3. Find $\displaystyle\int t^2 e^t dt$.

Example 4. Evaluate $\int e^x \sin x dx$.

Theorem 7.1.2 (Formula for Definite Integration by Parts)**.** *If f and g are differentiable on (a, b) and f' and g' are continuous, then*

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)dx.$$

Example 5. Calculate $\int_0^1 \tan^{-1} x dx$.

Example 6. Prove the reduction formula

$$\int \sin^n x dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x dx$$

where $n \geq 2$ is an integer.

7.2 Trigonometric Integrals

Example 1. Evaluate $\int \cos^3 x dx$.

Example 2. Find $\int \sin^5 x \cos^2 x dx$.

Remark 1. Sometimes it is easier to use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \text{and} \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

to evaluate an integral.

Example 3. Evaluate $\int_0^{\pi} \sin^2 x dx$.

Example 4. Find $\int \sin^4 x dx$.

Example 5. Evaluate $\int \tan^6 x \sec^4 x dx$.

Example 6. Find $\int \tan^5 \theta \sec^7 \theta d\theta$.

Example 7. Find $\int \tan^3 x dx$.

Example 8. Find $\int \sec^3 x dx$.

Remark 2. To evaluate the integrals (a) $\int \sin mx \cos nx dx$, (b) $\int \sin mx \sin nx dx$, or (c) $\int \cos mx \cos nx dx$, use the corresponding identity:

$$\text{(a) } \sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$
$$\text{(b) } \sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\text{(c) } \cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)].$$

Example 9. Evaluate $\displaystyle\int \sin 4x \cos 5x dx$.

7.3 Trigonometric Substitution

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\ 0 \le \theta \le \dfrac{\pi}{2}$ or $\pi \le \theta \le \dfrac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Example 1. Evaluate $\displaystyle\int \frac{\sqrt{9 - x^2}}{x^2} dx$.

Example 2. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Example 3. Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}dx$.

Example 4. Find $\int \frac{x}{\sqrt{x^2+4}} dx$.

Example 5. Evaluate $\int \frac{dx}{\sqrt{x^2-a^2}}$, where $a > 0$.

Example 6. Find $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}dx$.

Example 7. Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}dx$.

7.4 Integration by Partial Fractions

Example 1. Find $\int \frac{x^3 + x}{x - 1} dx$.

Example 2. Evaluate $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$.

Example 3. Find $\displaystyle\int \frac{dx}{x^2 - a^2}$, where $a \neq 0$.

Example 4. Find $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$.

Theorem 7.4.1.

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C.$$

Example 5. Evaluate $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x} dx.$

Example 6. Evaluate $\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} dx.$

Example 7. Write out the form of the partial fraction decomposition of the function

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}.$$

Example 8. Evaluate $\displaystyle\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx.$

Example 9. Evaluate $\int \frac{\sqrt{x+4}}{x} dx$.

7.5 Strategy for Integration

Example 1. $\int \frac{\tan^3 x}{\cos^3 x} dx.$

Example 2. $\int e^{\sqrt{x}} dx.$

Example 3. $\displaystyle\int \frac{x^5+1}{x^3-3x^2-10x}dx.$

Example 4. $\displaystyle\int \frac{dx}{x\sqrt{\ln x}}$.

Example 5. $\displaystyle\int \sqrt{\frac{1-x}{1+x}}dx$.

7.6 Integration Using Tables and CAS's

Example 1. The region bounded by the curves $y = \arctan x$, $y = 0$, and $x = 1$ is rotated about the y-axis. Find the volume of the resulting solid.

Example 2. Use the Table of Integrals to find $\int \frac{x^2}{\sqrt{5-4x^2}}dx$.

Example 3. Use the Table of Integrals to evaluate $\int x^3 \sin x dx$.

Example 4. Use the Table of Integrals to find $\int x\sqrt{x^2 + 2x + 4}dx$.

Example 5. Use a computer algebra system to find $\int x\sqrt{x^2+2x+4}dx$.

Example 6. Use a CAS to evaluate $\int x(x^2+5)^8dx$.

Example 7. Use a CAS to find $\int \sin^5 x \cos^2 x dx$.

7.7 Approximate Integration

Theorem 7.7.1 (Midpoint Rule).

$$\int_a^b f(x)dx \approx M_n = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \textit{midpoint of } [x_{i-1}, x_i].$$

Theorem 7.7.2 (Trapezoidal Rule).

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

where $\Delta x = (b-a)/n$ *and* $x_i = a + i\Delta x$.

Example 1. Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with $n = 5$ to approximate the integral $\int_1^2 (1/x)dx$.

Theorem 7.7.3 (Error Bounds). *Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then*

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad \textit{and} \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2}.$$

Example 2. How large should we take n in order to guarantee that the Trapezoidal and Midpoint Rule approximations for $\int_1^2 (1/x)dx$ are accurate to within 0.0001?

Example 3. (a) Use the Midpoint Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.

(b) Give an upper bound for the error involved in this approximation.

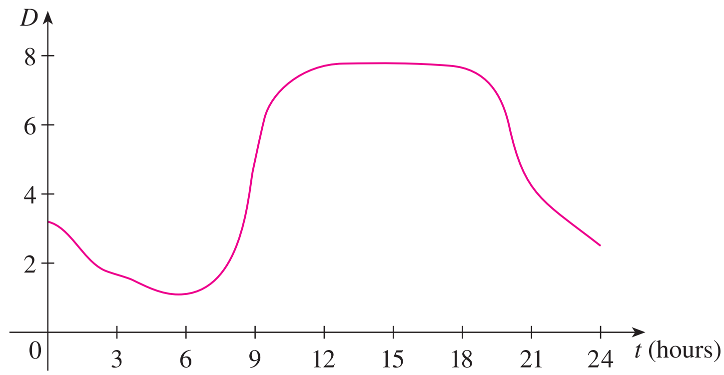
Theorem 7.7.4 (Simpson's Rule)**.**

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n *is even and* $\Delta x = (b-a)/n$.

Example 4. Use Simpson's Rule with $n = 10$ to approximate $\int_1^2 (1/x)dx$.

Example 5. The figure shows data traffic on the link from the United States to SWITCH, the Swiss academic and research network, on February 10, 1998. $D(t)$ is the data throughput, measured in megabits per second (Mb/s). Use Simpson's Rule to estimate the total amount of data transmitted on the link from midnight to noon on that day.



Theorem 7.7.5 (Error Bound for Simpson's Rule)**.** *Suppose that* $|f^{(4)}(x)| \leq K$ *for* $a \leq x \leq b$*. If* E_S *is the error involved in using Simpson's Rule, then*

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

Example 6. How large should we take n in order to guarantee that the Simpson's Rule approximation for $\int_1^2 (1/x)dx$ is accurate to within 0.0001?

Example 7. (a) Use Simpson's Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.

(b) Estimate the error involved in this approximation.

7.8 Improper Integrals

Definition 7.8.1 (Definition of an Improper Integral of Type 1).

(a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx.$$

In part (c) any real number a can be used.

Example 1. Determine whether the integral $\int_1^\infty (1/x)dx$ is convergent or divergent.

Example 2. Evaluate $\int_{-\infty}^{0} xe^x dx$.

Example 3. Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$.

Example 4. For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p} dx$$

convergent?

Definition 7.8.2 (Definition of an Improper Integral of Type 2)**.**

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)dx$ is called convergent if the corresponding limit exists and divergent if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Example 5. Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}dx$.

Example 6. Determine whether $\int_0^{\pi/2} \sec x dx$ converges or diverges.

Example 7. Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

Example 8. $\int_0^1 \ln x dx$.

Theorem 7.8.1 (Comparison Theorem)**.** *Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.*

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Example 9. Show that $\displaystyle\int_0^\infty e^{-x^2}dx$ is convergent.

Example 10. Determine whether $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}dx$ converges or diverges.

Chapter 8

Further Applications of Integration

8.1 Arc Length

Theorem 8.1.1 (The Arc Length Formula)**.** *If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \le x \le b$, is*

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \int_a^b \sqrt{1 + [f'(x)]^2}dx$$

where P_i is the point $(x_i, f(x_i))$.

Proof. Let $\Delta y_i = y_i - y_{i-1}$. By the Mean Value Theorem, there is a number x_i^* between x_{i-1} and x_i such that

$$\begin{aligned} f(x_i) - f(x_{i-1}) &= f'(x_i^*)(x_i - x_{i-1}) \\ \Delta y_i &= f'(x_i^*)\Delta x. \end{aligned}$$

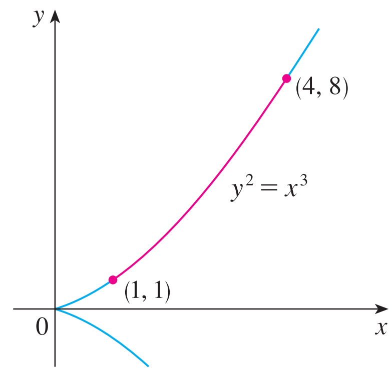
Therefore,

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x)^2 + (\Delta y_i)^2} \\ &= \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2} \\ &= \sqrt{1 + [f'(x_i^*)]^2}\sqrt{(\Delta x)^2} \\ &= \sqrt{1 + [f'(x_i^*)]^2}\Delta x. \end{aligned}$$

Hence

$$\lim_{n\to\infty}\sum_{i=1}^{n}|P_{i-1}P_i| = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+[f'(x_i^*)]^2}\Delta x = \int_a^b \sqrt{1+[f'(x)]^2}dx. \qquad \square$$

Example 1. Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1,1)$ and $(4,8)$. (See the figure.)



Example 2. Find the length of the arc of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

Example 3. (a) Set up an integral for the length of the arc of the hyperbola $xy = 1$ from the point $(1, 1)$ to the point $(2, \frac{1}{2})$.

(b) Use Simpson's Rule with $n = 10$ to estimate the arc length.

Theorem 8.1.2. *If a* <u>*smooth*</u> *curve C (a curve that has a continuous derivative) has the equation $y = f(x)$, $a \leq x \leq b$, then $s(x)$, the distance along C from the initial point $(a, f(a))$ to the point $(x, f(x))$, is called the* <u>*arc length function*</u> *and is given by*

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}dt.$$

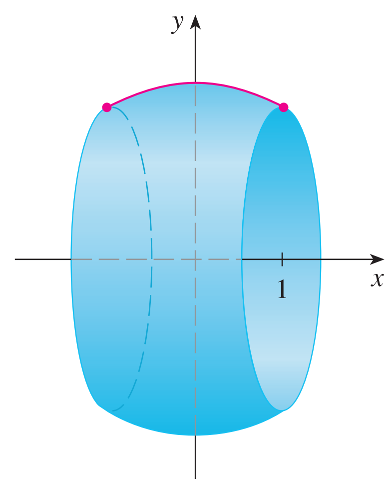
Example 4. Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $(1, 1)$ as the starting point.

8.2 Area of a Surface of Revolution

Definition 8.2.1. In the case where f is positive and has a continuous derivative, we define the surface area of the surface obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis as

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\Delta x = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}dx.$$

Example 1. The curve $y = \sqrt{4-x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis. (The surface is a portion of a sphere of radius 2. See the figure.)



Example 2. The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the area of the resulting surface.

Example 3. Find the area of the surface generated by rotating the curve $y = e^x$, $0 \le x \le 1$, about the x-axis.

8.3 Applications to Physics and Engineering

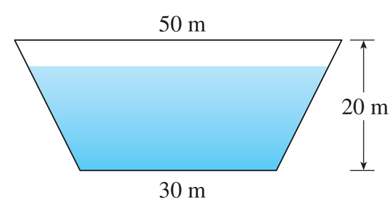
Definition 8.3.1. In general, the hydrostatic force exerted on a thin plate with area A square meters submerged in a fluid with density ρ kilograms per cubic meter at a depth d meters below the surface of the fluid is

$$F = mg = \rho g A d$$

where m is the mass and g is the acceleration due to gravity. The pressure P (in pascals) on the plate is defined to be the force per unit area:

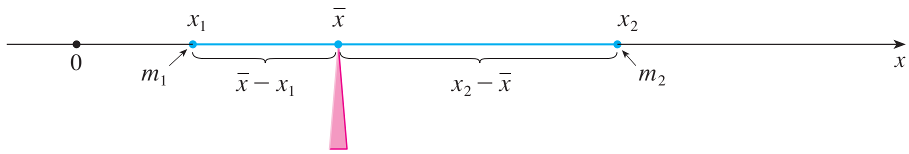
$$P = \frac{F}{A} = \rho g d.$$

Example 1. A dam has the shape of the trapezoid shown in the figure. The height is 20 m and the width is 50 m at the top and 30 m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam.



Example 2. Find the hydrostatic force on one end of a cylindrical drum with radius 3 ft if the drum is submerged in water 10 ft deep.

Definition 8.3.2. In general, for a system of n particles with masses $m_1, m_2, \dots, m_n$ located at the points $x_1, x_2, \dots, x_n$ on the x-axis,



the center of mass $\bar{x}$ is the point on which a thin plate of any given shape balances horizontally, and can be shown to be

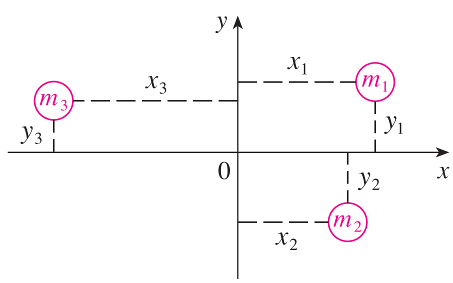
$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{m},$$

where $m_i x_i$ are called the moments of the masses m_i and $m = \sum m_i$ is the total mass of the system.
The sum of the individual moments

$$M = \sum_{i=1}^{n} m_i x_i$$

is called the moment of the system about the origin.

Definition 8.3.3. In general, for a system of n particles with masses $m_1, m_2, \dots, m_n$ located at the points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ in the xy-plane



we define the moment of the system about the y-axis to be

$$M_y = \sum_{i=1}^{n} m_i x_i$$

and the moment of the system about the x-axis to be

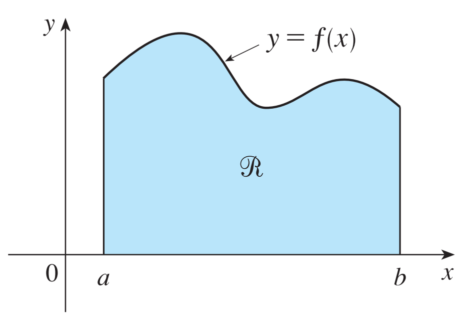
$$M_x = \sum_{i=1}^{n} m_i y_i.$$

The coordinates $(\bar{x}, \bar{y})$ of the center of mass are given by

$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}.$$

Example 3. Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points $(-1,1)$, $(2,-1)$, and $(3,2)$, respectively.

Definition 8.3.4. The center of mass of a lamina (a flat plate) with uniform density ρ and area A that occupies a region $\mathscr{R}$ of the plane



is called the centroid of $\mathscr{R}$ and is located at the point $(\bar{x}, \bar{y})$, where

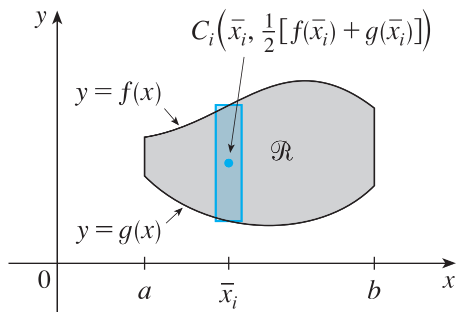
$$\bar{x} = \frac{1}{A}\int_a^b xf(x)dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2 dx.$$

Remark 1. The symmetry principle says that if $\mathscr{R}$ is symmetric about a line l, then the centroid of $\mathscr{R}$ lies on l.

Example 4. Find the center of mass of a semicircular plate of radius r.

Example 5. Find the centroid of the region bounded by the curves $y = \cos x$, $y = 0$, $x = 0$, and $x = \pi/2$.

Theorem 8.3.1. *If the region $\mathscr{R}$ lies between two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$,*



then the centroid of $\mathscr{R}$ is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]dx$$
$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\{[f(x)]^2 - [g(x)]^2\}dx.$$

Example 6. Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$.

Theorem 8.3.2 (Theorem of Pappus)**.** *Let $\mathscr{R}$ be a plane region that lies entirely on one side of a line l in the plane. If $\mathscr{R}$ is rotated about l, then the volume of the resulting solid is the product of the area A of $\mathscr{R}$ and the distance d traveled by the centroid of $\mathscr{R}$.*

Example 7. A torus is formed by rotating a circle of radius r about a line in the plane of the circle that is a distance R $(> r)$ from the center of the circle. Find the volume of the torus.

8.4 Applications to Economics and Biology

Definition 8.4.1. The consumer surplus for a commodity is defined as

$$\int_0^X [p(x) - P]dx$$

where $p(x)$ is the demand function, and P is the current selling price for the amount of the commodity X that can currently be sold.

Example 1. The demand for a product, in dollars, is

$$p = 1200 - 0.2x - 0.0001x^2.$$

Find the consumer surplus when the sales level is 500.

Definition 8.4.2. The cardiac output of the heart is the volume of blood pumped by the heart per unit time, that is, the rate of flow into the aorta. It is given by

$$F = \frac{A}{\int_0^T c(t)dt}$$

where A is the amount of dye injected into the right atrium, $[0, T]$ is the time interval until the dye has cleared, and $c(t)$ is the concentration of the dye at time t.

Example 2. A 5-mg bolus of dye is injected into a right atrium. The concentration of the dye (in milligrams per liter) is measured in the aorta at one-second intervals as shown in the table. Estimate the cardiac output.

t	$c(t)$
0	0
1	0.4
2	2.8
3	6.5
4	9.8
5	8.9
6	6.1
7	4.0
8	2.3
9	1.1
10	0

8.5 Probability

Definition 8.5.1. The probability density function f of a continuous random variable X (a quantity whose values range over an interval of real numbers) is given by:

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

where $f(x) \geq 0$ for all x and

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

Example 1. Let $f(x) = 0.006x(10 - x)$ for $0 \leq x \leq 10$ and $f(x) = 0$ for all other values of x.

(a) Verify that f is a probability density function.

(b) Find $P(4 \leq X \leq 8)$

Example 2. Phenomena such as waiting times and equipment failure times are commonly modeled by exponentially decreasing probability density functions. Find the exact form of such a function.

Definition 8.5.2. In general, the mean of any probability density function f is defined to be
$$\mu = \int_{-\infty}^{\infty} xf(x)dx.$$

Example 3. Find the mean of the exponential distribution of Example 2:

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ ce^{-ct} & \text{if } t \geq 0. \end{cases}$$

Example 4. Suppose the average waiting time for a customer's call to be answered by a company representative is five minutes.

(a) Find the probability that a call is answered during the first minute, assuming that an exponential distribution is appropriate.

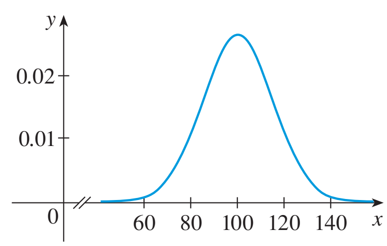
(b) Find the probability that a customer waits more than five minutes to be answered.

Definition 8.5.3. When random phenomena are modeled by a normal distribution this means that the probability density function of the random variable X is a member of the family of functions

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

where the positive constant σ is called the standard deviation (a measure of how spread out the values of X are).

Example 5. Intelligence Quotient (IQ) scores are distributed normally with mean 100 and standard deviation 15. (The figure shows the corresponding probability density function.)



(a) What percentage of the population has an IQ score between 85 and 115?

(b) What percentage of the population has an IQ above 140?

Chapter 9

Differential Equations

9.1 Modeling with Differential Equations

Definition 9.1.1. In general, a differential equation is an equation that contains an unknown function and one or more of its derivatives. The order of a differential equation is the order of the highest derivative that occurs in the equation. A function f is called a solution of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

Example 1. Show that every member of the family of functions

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$.

Example 2. Find a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$ that satisfies the initial condition $y(0) = 2$.

9.2 Direction Fields and Euler's Method

Definition 9.2.1. In general, suppose we have a first-order differential equation of the form

$$y' = F(x, y)$$

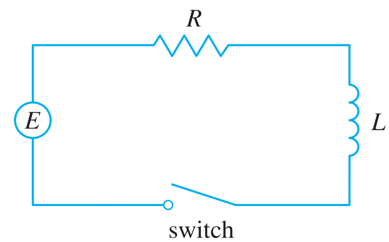
where $F(x, y)$ is some expression in x and y. If we draw short line segments with slope $F(x, y)$ at several points (x, y), the result is called a direction field (or slope field).

Example 1.

(a) Sketch the direction field for the differential equation $y' = x^2 + y^2 - 1$.

(b) Use part (a) to sketch the solution curve that passes through the origin.

Example 2. Suppose that in the simple circuit of the figure the resistance is 12 Ω, the inductance is 4 H, and a battery gives a constant voltage of 60 V.



(a) Draw a direction field for

$$L\frac{dI}{dt} + RI = E(t)$$

with these values.

(b) What can you say about the limiting value of the current?

(c) Identify any equilibrium solutions.

(d) If the switch is closed when $t = 0$ so the current starts with $I(0) = 0$, use the direction field to sketch the solution curve.

Theorem 9.2.1 (Euler's Method)**.** *Approximate values for the solution of the initial-value problem* $y' = F(x, y)$, $y(x_0) = y_0$ *with step size* h*, at* $x_n = x_{n-1} + h$*, are*

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \qquad n = 1, 2, 3, \ldots.$$

Example 3. Use Euler's method with step size 0.1 to construct a table of approximate values for the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1.$$

Example 4. In Example 2 we discussed a simple electric circuit with resistance 12 Ω, inductance 4 H, and a battery with voltage 60 V. If the switch is closed when $t = 0$, we modeled the current I at time t by the initial-value problem

$$\frac{dI}{dt} = 15 - 3I \qquad I(0) = 0.$$

Estimate the current in the circuit half a second after the switch is closed.

9.3 Separable Equations

Definition 9.3.1. A separable equation is a first-order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y. In other words, it can be written in the form

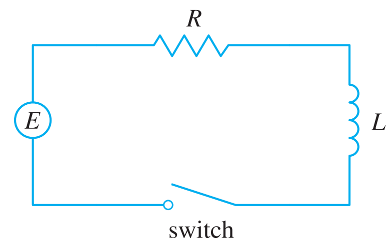
$$\frac{dy}{dx} = g(x)g(y).$$

Example 1. (a) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

Example 2. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$.

Example 3. Solve the equation $y' = x^2 y$.

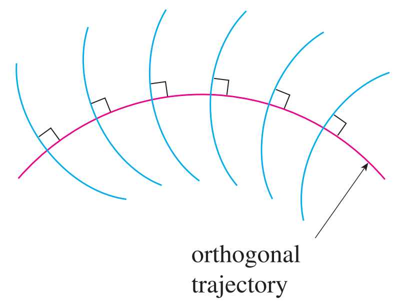


Example 4. In Section 9.2 we modeled the current $I(t)$ in the electric circuit shown in the figure by the differential equation

$$L\frac{dI}{dt} + RI = E(t).$$

Find an expression for the current in a circuit where the resistance is 12 V, the inductance is 4 H, a battery gives a constant voltage of 60 V, and the switch is turned on when $t = 0$. What is the limiting value of the current?

Definition 9.3.2. An orthogonal trajectory of a family of curves is a curve that intersects each curve of the family orthogonally, that is, at right angles (see the figure).



Example 5. Find the orthogonal trajectories of the family of curves $x = ky^2$, where k is an arbitrary constant.

Example 6. A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

9.4 Models for Population Growth

Definition 9.4.1. In general, if $P(t)$ is the value of a quantity y at time t and if the rate of change of P with respect to t is proportional to its size $P(t)$ at any time, then

$$\frac{dP}{dt} = kP$$

where k is a constant. This equation is sometimes called the law of natural growth.

Theorem 9.4.1. *The solution of the initial-value problem*

$$\frac{dP}{dt} = kP \qquad P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}.$$

Proof. The law of natural growth is a separable differential equation, so

$$\begin{aligned}
\frac{dP}{dt} &= kP \\
\int \frac{dP}{P} &= \int k dt \\
\ln|P| &= kt + C \\
|P| &= e^{kt+C} = e^C e^{kt} \\
P &= Ae^{kt},
\end{aligned}$$

where A $(= \pm e^C$ or $0)$ is an arbitrary constant. Since $P(0) = A$, $P(t) = P_0 e^{kt}$. □

Definition 9.4.2. The model for population growth known as the logistic differential equation is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

where M is the carrying capacity, the maximum population that the environment is capable of sustaining in the long run.

Example 1. Draw a direction field for the logistic equation with $k = 0.08$ and carrying capacity $M = 1000$. What can you deduce about the solutions?

Theorem 9.4.2. *The solution to the logistic equation is*

$$P(t) = \frac{M}{1 + Ae^{-kt}} \qquad \text{where } A = \frac{M - P_0}{P_0}.$$

Proof. The logistic equation is separable, so using partial fractions, we get

$$\begin{aligned}
\frac{dP}{dt} &= kP\left(1 - \frac{P}{M}\right) \\
\int \frac{dP}{P(1 - P/M)} &= \int k\,dt \\
\int \frac{M}{P(M - P)}dP &= \int k\,dt \\
\int \left(\frac{1}{P} + \frac{1}{M - P}\right) dP &= \int k\,dt \\
\ln|P| - \ln|M - P| &= kt + C \\
\ln\left|\frac{M - P}{P}\right| &= -kt - C \\
\left|\frac{M - P}{P}\right| &= e^{-kt-C} = e^{-C}e^{-kt} \\
\frac{M - P}{P} &= Ae^{-kt} \\
\frac{M}{P} - 1 &= Ae^{-kt} \\
\frac{M}{P} &= 1 + Ae^{-kt} \\
P &= \frac{M}{1 + Ae^{-kt}},
\end{aligned}$$

where $A = \pm e^{-C}$. If $t = 0$, we have

$$\frac{M - P_0}{P_0} = Ae^0 = A. \qquad \square$$

Example 2. Write the solution of the initial-value problem

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) \qquad P(0) = 100$$

and use it to find the population sizes $P(40)$ and $P(80)$. At what time does the population reach 900?

Example 3. In the 1930s the biologist G. F. Gause conducted an experiment with the protozoan *Paramecium* and used a logistic equation to model his data. The table gives his daily count of the population of protozoa. He estimated the initial relative growth rate to be 0.7944 and the carrying capacity to be 64.

t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57

Find the exponential and logistic models for Gause's data. Compare the predicted values with the observed values and comment on the fit.

9.5 Linear Equations

Definition 9.5.1. A first-order linear differential equation is one that can be put into the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions on a given interval.

Theorem 9.5.1. *To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $I(x) = e^{\int P(x)dx}$ and integate both sides.*

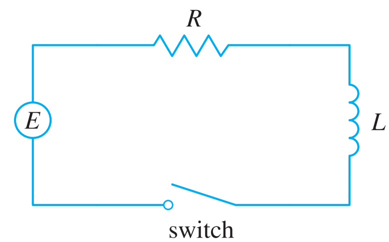
Example 1. Solve the differential equation $\dfrac{dy}{dx} + 3x^2y = 6x^2$.

Example 2. Find the solution of the initial-value problem

$$x^2y' + xy = 1 \qquad x > 0 \qquad y(1) = 2.$$

Example 3. Solve $y' + 2xy = 1$.

Example 4. Suppose that in the simple circuit of the figure the resistance is 12 V and the inductance is 4 H. If a battery gives a constant voltage of 60 V and the switch is closed when $t = 0$ so the current starts with $I(0) = 0$, find



(a) $I(t)$,

(b) the current after 1 second, and

(c) the limiting value of the current.

Example 5. Suppose that the resistance and inductance remain as in Example 4 but, instead of the battery, we use a generator that produces a variable voltage of $E(t) = 60\sin 30t$ volts. Find $I(t)$.

9.6 Predator-Prey Systems

Definition 9.6.1. The equations

$$\frac{dR}{dt} = kR - aRW \qquad \frac{dW}{dt} = -rW + bRW$$

are known as the predator-prey equations, or the Lotka-Volterra equations. A solution of this system of equations is a pair of functions $R(t)$ and $W(t)$ that describe the populations of prey and predators as functions of time.

Example 1. Suppose that populations of rabbits and wolves are described by the Lotka-Volterra equations with $k = 0.08$, $a = 0.001$, $r = 0.02$, and $b = 0.00002$. The time t is measured in months.

(a) Find the constant solutions (called the equilibrium solutions) and interpret the answer.

(b) Use the system of differential equations to find an expression for dW/dR.

(c) Draw a direction field for the resulting differential equation in the RW-plane. Then use that direction field to sketch some solution curves.

(d) Suppose that, at some point in time, there are 1000 rabbits and 40 wolves. Draw the corresponding solution curve and use it to describe the changes in both population levels.

(e) Use part (d) to make sketches of R and W as functions of t.

Chapter 10

Parametric Equations and Polar Coordinates

10.1 Curves Defined by Parametric Equations

Definition 10.1.1. Suppose that x and y are both given as functions of a third variable t (called a parameter) by the equations

$$x = f(t) \qquad y = g(t)$$

(called parametric equations). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a parametric curve.

Example 1. Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1.$$

Definition 10.1.2. In general, the curve with parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \leq t \leq b$$

has initial point $(f(a), g(a))$ and terminal point $(f(b), g(b))$.

Example 2. What curve is represented by the following parametric equations?

$$x = \cos t \qquad y = \sin t \qquad 0 \leq t \leq 2\pi.$$

Example 3. What curve is represented by the given parametric equations?

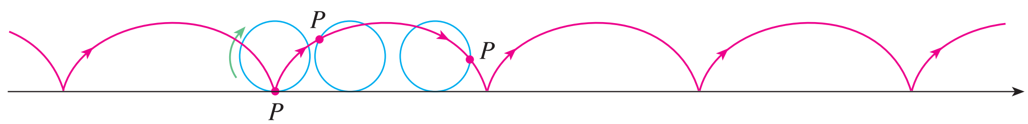
$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leq t \leq 2\pi.$$

Example 4. Find parametric equations for the circle with center (h, k) and radius r.

Example 5. Sketch the curve with parametric equations $x = \sin t$, $y = \sin^2 t$.

Example 6. Use a graphing device to graph the curve $x = y^4 - 3y^2$.

Example 7. The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a cycloid (see the figure). If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.



Example 8. Investigate the family of curves with parametric equations

$$x = a + \cos t \qquad y = a \tan t + \sin t.$$

What do these curves have in common? How does the shape change as a increases?

10.2 Calculus with Parametric Curves

Theorem 10.2.1. *Suppose f and g are differentiable functions. Then for a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x, we have*

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{if } \frac{dx}{dt} \neq 0.$$

Proof. Since y is a differentiable function of x, we have, by the Chain Rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Then if $\frac{dx}{dt} \neq 0$ we can divide by it, so

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$ □

Theorem 10.2.2. *Suppose f and g are differentiable functions. Then for a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x, we have*

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \qquad \text{if } \frac{dx}{dt} \neq 0.$$

Proof. By the previous theorem,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \qquad \text{if } \frac{dx}{dt} \neq 0.$$ □

Example 1. A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

(a) Show that C has two tangents at the point $(3, 0)$ and find their equations

(b) Find the points on C where the tangent is horizontal or vertical.

(c) Determine where the curve is concave upward or downward.

(d) Sketch the curve.

Example 2.

(a) Find the tangent to the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ at the point where $\theta = \pi/3$.

(b) At what points is the tangent horizontal? When is it vertical?

Theorem 10.2.3. *If a curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$, then the area under the curve is given by*

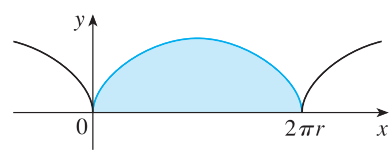
$$A = \int_\alpha^\beta g(t)f'(t)dt \qquad \left[or \int_\beta^\alpha g(t)f'(t)dt \right].$$

Proof. Since the area under the curve $y = F(x)$ from a to b is $A = \int_a^b F(x)dx$, we can use the Substitution Rule for Definite Integrals with $y = g(t)$ and $dx = f'(t)dt$ to get

$$A = \int_a^b y dx = \int_\alpha^\beta g(t)f'(t)dt.$$ □

Example 3. Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta).$$



(See the figure.)

Theorem 10.2.4. *If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is*

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example 4. (a) Use the representation of the unit circle given by

$$x = \cos t \qquad y = \sin t \qquad 0 \leq t \leq 2\pi$$

to find its arc length.

(b) Use the representation of the unit circle given by

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leq t \leq 2\pi$$

to find its arc length.

Example 5. Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

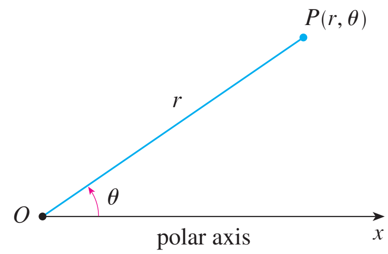
Theorem 10.2.5. *Suppose a curve C is given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f', g' are continuous, $g'(t) \geq 0$, is rotated about the x-axis. If C is traversed exactly once as t increases from α to β, then the area of the resulting surface is given by*

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example 6. Show that the surface area of a sphere of radius r is $4\pi r^2$.

10.3 Polar Coordinates

Definition 10.3.1. The polar coordinate system consists of a point called the pole (or origin) O, a ray starting at the pole called the polar axis, and other points P represented by (r, θ) where r is the distance from O to P and θ is the angle (usually measured in radians) between the polar axis and the line OP as in the figure. r, θ are called polar coordinates of P.



Example 1. Plot the points whose polar coordinates are given.

(a) $(1, 5\pi/4)$

(b) $(2, 3\pi)$

(c) $(2, -2\pi/3)$

(d) $(-3, 3\pi/4)$

Theorem 10.3.1. *If the point P has Cartesian coordinates* (x, y) *and polar coordinates* (r, θ), *then*

$$x = r\cos\theta \qquad y = r\sin\theta$$

and

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}.$$

Example 2. Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

Example 3. Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

Example 4. What curve is represented by the polar equation $r = 2$?

Example 5. Sketch the polar curve $\theta = 1$.

Example 6. (a) Sketch the curve with polar equation $r = 2\cos\theta$.

(b) Find a Cartesian equation for this curve.

Example 7. Sketch the curve $r = 1 + \sin\theta$.

Example 8. Sketch the curve $r = \cos 2\theta$.

Theorem 10.3.2. *The slope of the tangent line to a polar curve $r = f(\theta)$ is*

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Proof. Regard θ as a parameter and write

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Then by Theorem 10.2.1 and the product rule, we have

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}. \qquad \square$$

Example 9.

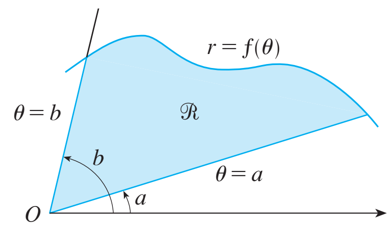
(a) For the cardioid $r = 1 + \sin\theta$ of Example 7, find the slope of the tangent line when $\theta = \pi/3$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

Example 10. Graph the curve $r = \sin(8\theta/5)$.

Example 11. Investigate the family of polar curves given by $r = 1 + c\sin\theta$. How does the shape change as c changes? (These curves are called limaçons, after a French word for snail, because of the shape of the curves for certain values of c.)

10.4 Areas and Lengths in Polar Coordinates



Theorem 10.4.1. *Let $\mathscr{R}$ be the region, illustrated in the figure, bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leq 2\pi$. The area A of the polar region $\mathscr{R}$ is*

$$A = \int_a^b \frac{1}{2} r^2 d\theta.$$

Example 1. Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Example 2. Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Example 3. Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

Theorem 10.4.2. *The length of a curve with polar equation $r = f(\theta)$, $a \leq \theta \leq b$, is*

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Proof. Regard θ as a parameter and write

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Then by the product rule, we have

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \qquad \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta.$$

Since $\cos^2\theta + \sin^2\theta = 1$,

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2,\end{aligned}$$

so

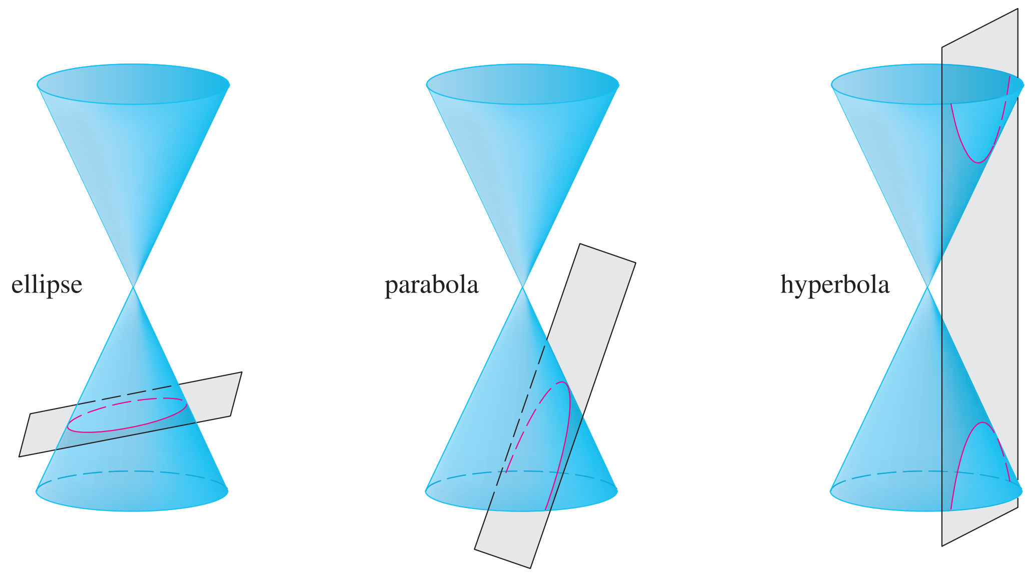
$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

□

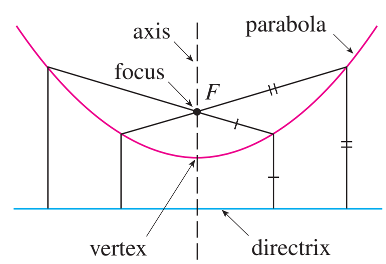
Example 4. Find the length of the cardioid $r = 1 + \sin\theta$.

10.5 Conic Sections

Definition 10.5.1. Parabolas, ellipses, and hyperbolas are called conic sections, or conics, because they result from intersecting a cone with a plane as shown in the figure.



Definition 10.5.2. A parabola is the set of points in a plane that are equidistant from a fixed point F (called the focus) and a fixed line (called the directrix). This definition is illustrated by the figure. Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the vertex. The line through the focus perpendicular to the directrix is called the axis of the parabola.



Theorem 10.5.1. *An equation of the parabola with focus* $(0, p)$ *and directrix* $y = -p$ *is*

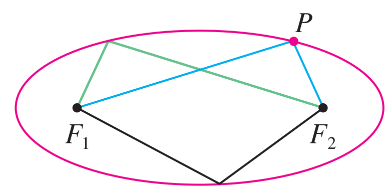
$$x^2 = 4py.$$

Theorem 10.5.2. *An equation of the parabola with focus* $(p, 0)$ *and directrix* $x = -p$ *is*

$$y^2 = 4px.$$

Example 1. Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

Definition 10.5.3. An ellipse is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant (see the figure). These two fixed points are called the foci (plural of focus).



Definition 10.5.4. If $(-c, 0)$ and $(c, 0)$ are the foci of an ellipse, the sum of the distances from a point on the ellipse to the foci are $2a > 0$, and $b^2 = a^2 - c^2$, then the points $(a, 0)$ and $(-a, 0)$ are called the vertices of ellipse and the line segment joining the vertices is called the major axis. The line segment joining $(0, b)$ and $(0, -b)$ is the minor axis.

Theorem 10.5.3. *The ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \geq b > 0$$

has foci $(\pm c, 0)$*, where* $c^2 = a^2 - b^2$*, and vertices* $(\pm a, 0)$*.*

Theorem 10.5.4. *The ellipse*

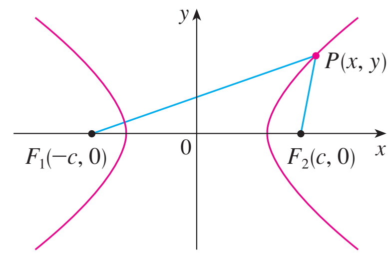
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \geq b > 0$$

has foci $(0, \pm c)$*, where* $c^2 = a^2 - b^2$*, and vertices* $(0, \pm a)$*.*

Example 2. Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

Example 3. Find an equation of the ellipse with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.

Definition 10.5.5. A hyperbola is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the foci) is a constant. This definition is illustrated in the figure.



Theorem 10.5.5. *The hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci $(\pm c, 0)$*, where* $c^2 = a^2 + b^2$*, vertices* $(\pm a, 0)$*, and asymptotes* $y = \pm(b/a)x$*.*

Theorem 10.5.6. *The hyperbola*

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has foci $(0, \pm c)$*, where* $c^2 = a^2 + b^2$*, vertices* $(0, \pm a)$*, and asymptotes* $y = \pm(a/b)x$*.*

Example 4. Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

Example 5. Find the foci and equation of the hyperbola with vertices $(0, \pm 1)$ and asymptote $y = 2x$.

Example 6. Find an equation of the ellipse with foci $(2, -2)$, $(4, -2)$, and vertices $(1, -2)$, $(5, -2)$.

Example 7. Sketch the conic $9x^2 - 4y^2 - 72x + 8y + 176 = 0$ and find its foci.

10.6 Conic Sections in Polar Coordinates

Theorem 10.6.1. *Let F be a fixed point (called the focus) and l be a fixed line (called the directrix) in a plane. Let e be a fixed positive number (called the eccentricity). The set of all points P in the plane such that*

$$\frac{|PF|}{|Pl|} = e$$

(that is, the ratio of the distance from F to the distance from l is the constant e) is a conic section. The conic is

(a) an ellipse if $e < 1$

(b) a parabola if $e = 1$

(c) a hyperbola if $e > 1$

Theorem 10.6.2. *A polar equation of the form*

$$r = \frac{ed}{1 \pm e\cos\theta} \qquad \text{or} \qquad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.

Example 1. Find a polar equation for a parabola that has its focus at the origin and whose directrix is the line $y = -6$.

Example 2. A conic is given by the polar equation

$$r = \frac{10}{3 - 2\cos\theta}.$$

Find the eccentricity, identify the conic, locate the directrix, and sketch the conic.

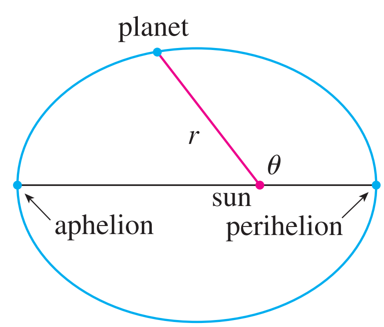
Example 3. Sketch the conic $r = \dfrac{12}{2 + 4\sin\theta}$.

Example 4. If the ellipse of Example 2 is rotated through an angle $\pi/4$ about the origin, find a polar equation and graph the resulting ellipse.

Theorem 10.6.3. *The polar equation of an ellipse with focus at the origin, semimajor axis a, eccentricity e, and directrix $x = d$ can be written in the form*

$$r = \frac{a(1-e^2)}{1+e\cos\theta}.$$

Definition 10.6.1. The positions of a planet that are closest to and farthest from the sun are called its perihelion and aphelion, respectively, and correspond to the vertices of the ellipse (see the figure). The distances from the sun to the perihelion and aphelion are called the perihelion distance and aphelion distance, respectively.



Theorem 10.6.4. *The perihelion distance from a planet to the sun is $a(1-e)$ and the aphelion distance is $a(1+e)$.*

Proof. If the sun is at the focus F, at perihelion we have $\theta = 0$, so

$$r = \frac{a(1-e^2)}{1+e\cos 0} = \frac{a(1-e)(1+e)}{1+e} = a(1-e).$$

Similarly, at aphelion $\theta = \pi$ and $r = a(1+e)$. □

Example 5. (a) Find an approximate polar equation for the elliptical orbit of the earth around the sun (at one focus) given that the eccentricity is about 0.017 and the length of the major axis is about 2.99×10^8 km.

(b) Find the distance from the earth to the sun at perihelion and at aphelion.

Chapter 11

Infinite Sequences and Series

11.1 Sequences

Definition 11.1.1. A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots.$$

The number a_1 is called the first term, a_2 is the second term, and in general a_n is the nth term.
A sequence can also be defined as a function whose domain is the set of positive integers. However, we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.
The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}.$$

Example 1. Some sequences can be defined by giving a formula for the nth term. In the following examples we give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that n doesn't have to start at 1.

(a)	$\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$	$a_n = \dfrac{n}{n+1}$	$\left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dots, \dfrac{n}{n+1}, \dots\right\}$
(b)	$\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}$	$a_n = \dfrac{(-1)^n(n+1)}{3^n}$	$\left\{-\dfrac{2}{3}, \dfrac{3}{9}, -\dfrac{4}{27}, \dfrac{5}{81}, \dots, \dfrac{(-1)^n(n+1)}{3^n}, \dots\right\}$
(c)	$\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$	$a_n = \sqrt{n-3}, n \geq 3$	$\left\{0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots\right\}$
(d)	$\left\{\cos\dfrac{n\pi}{6}\right\}_{n=0}^{\infty}$	$a_n = \cos\dfrac{n\pi}{6}, n \geq 0$	$\left\{1, \dfrac{\sqrt{3}}{2}, \dfrac{1}{2}, 0, \dots, \cos\dfrac{n\pi}{6}, \dots\right\}$

Example 2. Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \dots\right\}$$

assuming that the pattern of the first few terms continues.

Example 3. Here are some sequences that don't have a simple defining equation.

(a) The sequence $\{p_n\}$, where p_n is the population of the world as of January 1 in the year n.

(b) If we let a_n be the digit in the nth decimal place of the number e, then $\{a_n\}$ is a well-defined sequence whose first few terms are

$$\{7, 1, 8, 2, 8, 1, 8, 2, 4, 5, \ldots\}.$$

(c) The Fibonacci sequence $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1 \qquad f_2 = 1 \qquad f_n = f_{n-1} + f_{n-2} \qquad n \geq 3.$$

Each term is the sum of the two preceding terms. The first few terms are

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits.

Definition 11.1.2. A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty}$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

Definition 11.1.3 (Precise Definition of the Limit of a Sequence)**.** A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon.$$

Theorem 11.1.1. *If* $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = a_n$ *when* n *is an integer, then* $\lim_{n\to\infty} a_n = L$.

Definition 11.1.4. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad a_n > M.$$

Theorem 11.1.2 (Limit Laws for Sequences)**.** *If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then*

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$
$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$
$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0.$$

Theorem 11.1.3 (Squeeze Theorem for Sequences)**.** *If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

Theorem 11.1.4. *If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Proof. Since $\lim_{n\to\infty} |a_n| = 0$,

$$\lim_{n\to\infty} -|a_n| = 0 = -\lim_{n\to\infty} |a_n| = 0.$$

But $-|a_n| \leq a_n \leq |a_n|$ for all n, so by the squeeze theorem for sequences, $\lim_{n\to\infty} a_n = 0$. □

Example 4. Find $\lim_{n\to\infty} \dfrac{n}{n+1}$.

Example 5. Is the sequence $a_n = \dfrac{n}{\sqrt{10+n}}$ convergent or divergent?

Example 6. Calculate $\lim\limits_{n\to\infty} \dfrac{\ln n}{n}$.

Example 7. Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Example 8. Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

Theorem 11.1.5. *If* $\lim_{n\to\infty} a_n = L$ *and the function* f *is continuous at* L*, then*

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Proof. Choose a particular n, say n_0. By the definition of a limit of a sequence, given $\varepsilon_1 > 0$ there exists an integer N, such that $|a_{n_0} - L| < \varepsilon_1$ for $n_0 > N$. Similarly, by the definition of continuity, the limit of f exists at L, so for $\varepsilon_2 > 0$ there exists $\varepsilon_1 > 0$ such that if $|a_{n_0} - L| < \varepsilon_1$ then $|f(a_{n_0}) - f(L)| < \varepsilon_2$. This is true for arbitrary $\varepsilon_2 > 0$, so $\lim_{n\to\infty} f(a_n) = f(L)$. □

Example 9. Find $\lim_{n\to\infty} \sin(\pi/n)$.

Example 10. Discuss the convergence of the sequence $a_n = n!/n^n$, where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.

Example 11. For what values of r is the sequence $\{r^n\}$ convergent?

Definition 11.1.5. A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is monotonic if it is either increasing or decreasing.

Example 12. Is the sequence $\left\{ \dfrac{3}{n+5} \right\}$ increasing or decreasing?

Example 13. Show that the sequence $a_n = \dfrac{n}{n^2+1}$ is decreasing.

Definition 11.1.6. A sequence $\{a_n\}$ is bounded above if there is a number M such that

$$a_n \leq M \qquad \text{for all } n \geq 1.$$

It is bounded below if there is a number m such that

$$m \leq a_n \qquad \text{for all } n \geq 1.$$

If it is bounded above and below, then $\{a_n\}$ is a bounded sequence.

Theorem 11.1.6 (Monotonic Sequence theorem)**.** *Every bounded, monotonic sequence is convergent.*

Example 14. Investigate the sequence $\{a_n\}$ defined by the recurrence relation

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{2}(a_n + 6) \qquad \text{for } n = 1, 2, 3, \ldots.$$

11.2 Series

Definition 11.2.1. In general, if we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an infinite series (or just a series) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n.$$

Definition 11.2.2. Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} = s.$$

The number s is called the sum of the series. If the sequence $\{s_n\}$ is divergent, then the series is called divergent.

Example 1. Find the sum of the series $\sum_{n=1}^{\infty} a_n$ if the sum of the first n terms of the series is

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}.$$

Example 2. Find the sum of the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

where each term is obtained from the preceding one by multiplying it by the common ratio r.

Example 3. Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots .$$

Example 4. Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

Example 5. A drug is administered to a patient at the same time every day. Suppose the concentration of the drug is C_n (measured in mg/mL) after the injection on the nth day. Before the injection the next day, only 30% of the drug remains in the bloodstream and the daily dose raises the concentration by 0.2 mg/mL.

(a) Find the concentration after three days.

(b) What is the concentration after the nth dose?

(c) What is the limiting concentration?

Example 6. Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.

Example 7. Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$.

Example 8. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

Example 9. Show that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

Theorem 11.2.1. *If the series* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. Since $\sum a_n$ is convergent, the sequence $\{s_n\}$ is convergent. Let $\lim_{n\to\infty} s_n = s$. Since $n-1 \to \infty$ as $n \to \infty$, we also have $\lim_{n\to\infty} s_{n-1} = s$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0. \qquad \square$$

Corollary 11.2.1 (Test for Divergence)**.** *If* $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Proof. If the series is not divergent, then it is convergent, and so $\lim_{n\to\infty} a_n = 0$ by Theorem 11.2.1. The result follows by contrapositive. $\square$

Example 10. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Theorem 11.2.2. *If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$, and*

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Example 11. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.

11.3 The Integral Test and Estimates of Sums

Theorem 11.3.1 (The Integral Test). *Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and $a_n = f(n)$. The the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words:*

(i) If $\displaystyle\int_1^{\infty} f(x)dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\displaystyle\int_1^{\infty} f(x)dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Example 1. Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Example 2. For what values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent? (This series is called the p-series.)

Example 3. Determine whether each series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$

Example 4. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Definition 11.3.1. The remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

is the error made when s_n, the sum of the first n terms, is used as an approximation to the total sum.

Theorem 11.3.2 (Remainder Estimate for the Integral Test)**.** *Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then*

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx.$$

Example 5. (a) Approximate the sum of the series $\sum 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

Corollary 11.3.1. *Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. Then*

$$s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq s_n + \int_{n}^{\infty} f(x)dx.$$

Example 6. Use Corollary 11.3.1 with $n = 10$ to estimate the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

11.4 The Comparison Tests

Theorem 11.4.1 (The Comparison Test)**.** *Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.*

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

Proof. (i) Let

$$s_n = \sum_{i=1}^{n} a_i \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{\infty} b_n$$

Since both series have positive terms, the sequences $\{s_n\}$ and $\{t_n\}$ are increasing ($s_{n+1} = s_n + a_{n+1} \ge s_n$). Also $t_n \to t$, so $t_n \le t$ for all n. Since $a_i \le b_i$, we have $s_n \le t_n$. Thus $s_n \le t$ for all n. This means that $\{s_n\}$ is increasing and bounded above and therefore converges by the Monotonic Sequence Theorem. Thus $\sum a_n$ converges.

(ii) If $\sum b_n$ is divergent, then $t_n \to \infty$ (since $\{t_n\}$ is increasing). But $a_i \ge b_i$ so $s_n \ge t_n$. Thus $s_n \to \infty$. Therefore $\sum a_n$ diverges. □

Example 1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges or diverges.

Example 2. Test the series $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ for convergence or divergence.

Theorem 11.4.2 (The Limit Comparison Test)**.** *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with positive terms. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c *is a finite number and* $c > 0$*, then either both series converge or both diverge.*

Proof. Let m and M be positive numbers such that $m < c < M$. Because a_n/b_n is close to c for large n, there is an integer N such that

$$m < \frac{a_n}{b_n} < M \qquad \text{when } n > N,$$

and so

$$mb_n < a_n < Mb_n \qquad \text{when } n > N.$$

If $\sum b_n$ converges, so does $\sum Mb_n$. Thus $\sum a_n$ converges by part (i) of the Comparison Test. If $\sum b_n$ diverges, so does $\sum mb_n$ and part (ii) of the Comparison Test shows that $\sum a_n$ diverges. □

Example 3. Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

Example 4. Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

Example 5. Use the sum of the first 100 terms to approximate the sum of the series $\sum 1/(n^3+1)$. Estimate the error involved in this approximation.

11.5 Alternating Series

Definition 11.5.1. An alternating series is a series whose terms are alternately positive and negative.

Theorem 11.5.1 (Alternating Series Test)**.** *If the alternating series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

$$\text{(i)}\ \ b_{n+1} \leq b_n \qquad \text{for all } n$$
$$\text{(ii)}\ \ \lim_{n\to\infty} b_n = 0$$

then the series is convergent.

Example 1. Determine whether the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

converges or diverges.

Example 2. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ converges or diverges.

Example 3. Test the series $\sum_{n-1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ for convergence or divergence.

Theorem 11.5.2 (Alternating Series Estimation Theorem)**.** *If* $s = \sum(-1)^{n-1}b_n$*, where* $b_n > 0$*, is the sum of an alternating series that satisfies*

$$(i)\ b_{n+1} \leq b_n \qquad and \qquad (ii)\ \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Example 4. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

11.6 Absolute Convergence, Ratio and Root Tests

Definition 11.6.1. A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

Example 1. Is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

absolutely convergent?

Example 2. Is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

absolutely convergent?

Definition 11.6.2. A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

Theorem 11.6.1. *If a series $\sum a_n$ is absolutely convergent, then it is convergent.*

Proof. Observe that the inequality

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

is true because $|a_n|$ is either a_n or $-a_n$. If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent, so $\sum 2|a_n|$ is convergent. Therefore, by the Comparison Test, $\sum(a_n + |a_n|)$ is convergent. Then

$$\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$$

is the difference of two convergent series and is therefore convergent. □

Example 3. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

Theorem 11.6.2 (The Ratio Test)**.**

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent (and therefore convergent).*

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ *or* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, *the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of* $\sum a_n$.

Example 4. Test the series $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ for absolute convergence.

Example 5. Test the convergence of the series $\sum_{n=1}^{\infty}\frac{n^n}{n!}$.

Theorem 11.6.3 (The Root Test)**.**

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent (and therefore convergent).*

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ *or* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$*, the Root Test is inconclusive.*

Example 6. Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

Definition 11.6.3. By a rearrangement of an infinite series $\sum a_n$ we mean a series obtained by simply changing the order of the terms.

Remark 1. If $\sum a_n$ is an absolutely convergent series with sum s, then any rearrangement of $\sum a_n$ has the same sum s.

Remark 2. If $\sum a_n$ is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\sum a_n$ that has a sum equal to r. For example, if we multiply the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2$$

by $\frac{1}{2}$, we get

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots = \frac{1}{2}\ln 2.$$

Then inserting zeros between the terms of this series gives

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots = \frac{1}{2}\ln 2,$$

and we can add this to the alternating harmonic series to get

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}\ln 2,$$

which is a rearrangement of the alternating harmonic series with a different sum.

11.7 Strategy for Testing Series

Example 1. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$.

Example 2. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$.

Example 3. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$.

Example 4. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{n^4+1}$.

Example 5. $\displaystyle\sum_{n=1}^{\infty} \frac{2^k}{k!}$.

Example 6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$.

11.8 Power Series

Definition 11.8.1. A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.
More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a power series in $(x-a)$ or a power series centered at a or a power series about a.

Example 1. For what values of x is the series $\sum_{n=0}^{\infty} n! x^n$ convergent?

Example 2. For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Example 3. Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

Theorem 11.8.1. *For a given power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$*, there are only three possibilities:*

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x - a| > R$.

Definition 11.8.2. The number R in case (iii) is called the radius of convergence of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii). The interval of convergence of a power series is the interval that consists of all values of x for which the series converges.

Example 4. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}.$$

Example 5. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}.$$

11.9 Representations of Functions as Power Series

Example 1. Express $1/(1+x^2)$ as the sum of a power series and find the interval of convergence.

Example 2. Find a power series representation for $1/(x+2)$.

Example 3. Find a power series representation of $x^3/(x+2)$.

Theorem 11.9.1. *If the power series* $\sum c_n(x-a)^n$ *has radius of convergence* $R > 0$*, then the function* f *defined by*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ *and*

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\displaystyle\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}.$$

The radii of convergence of the power series in Equations (i) and (ii) are both R*.*

Example 4. Find the derivative of the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

Example 5. Express $1/(1-x)^2$ as a power series using differentiation. What is the radius of convergence?

Example 6. Find a power series representation for $\ln(1+x)$ and its radius of convergence.

Example 7. Find a power series representation for $f(x) = \tan^{-1} x$.

Example 8. (a) Evaluate $\int [1/(1+x^7)]dx$ as a power series.

(b) Use part (a) to approximate $\int_0^{0.5} [1/(1+x^7)]dx$ correct to within 10^{-7}.

11.10 Taylor and Maclaurin Series

Theorem 11.10.1. *If f has a power series representation (expansion) at a, that is, if*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Definition 11.10.1. The Taylor series of the function f at a (or about a or centered at a) is

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots . \end{aligned}$$

For the special case $a = 0$ the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots ,$$

which we call the Maclaurin Series.

Example 1. Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Theorem 11.10.2. *If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a, R_n is the remainder of the Taylor series, and*

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Theorem 11.10.3 (Taylor's Inequality)**.** *If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality*

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \le d.$$

Example 2. Prove that e^x is equal to the sum of its Maclaurin series.

Example 3. Find the Taylor series $f(x) = e^x$ at $a = 2$.

Example 4. Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

Example 5. Find the Maclaurin series for $\cos x$.

Example 6. Find the Maclaurin series for the function $f(x) = x \cos x$.

Example 7. Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\pi/3$.

Example 8. Find the Maclaurin series for $f(x) = (1 + x)^k$, where k is any real number.

Theorem 11.10.4 (The Binomial Series)**.** *If k is any real number and $|x| < 1$, then*

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

where the coefficients

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

are called the binomial coefficients.

Example 9. Find the Maclaurin series for the function $f(x) = \dfrac{1}{\sqrt{4-x}}$ and its radius of convergence.

Example 10. Find the sum of the series $\frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots$.

Example 11. (a) Evaluate $\int e^{-x^2}dx$ as an infinite series.

(b) Evaluate $\int_0^1 e^{-x^2}dx$ correct to within an error of 0.001.

Example 12. Evaluate $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.

Example 13. Find the first three nonzero terms in the Maclaurin series for

(a) $e^x \sin x$

(b) $\tan x$

11.11 Applications of Taylor Polynomials

Example 1. (a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

(b) How accurate is this approximation when $7 \leq x \leq 9$?

Example 2. (a) What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when $-0.3 \leq x \leq 0.3$? Use this approximation to find $\sin 12^\circ$ correct to six decimal places.

(b) For what values of x is this approximation accurate to within 0.00005?

Example 3. In Einstein's theory of special relativity the mass of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of an object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0c^2.$$

(a) Show that when v is very small compared with c, this expression for K agrees with classical Newtonian physics: $K = \frac{1}{2}m_0v^2$.

(b) Use Taylor's Inequality to estimate the difference in these expressions for K when $|v| \leq 100$ m/s.

Index

Bibliography

[1] Stewart, James. *Calculus: Early Transcendentals.* Boston, MA, USA: Cengage Learning, 2016. Print.