Multivariable Calculus Homework #3

Replace this text with your name

Due: Replace this text with a due date

Exercise (14.1.8)**.** A company makes three sizes of cardboard boxes: small, medium, and large. It costs \$2.50 to make a small box, \$4.00 for a medium box, and \$4.50 for a large box. Fixed costs are \$8000.

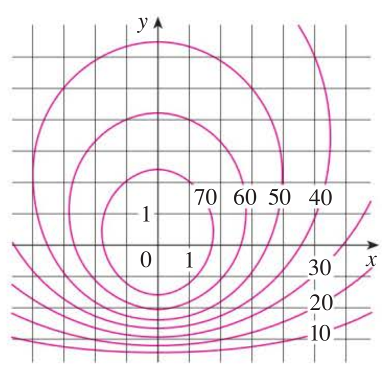
(a) Express the cost of making x small boxes, y medium boxes, and z large boxes as a function of three variables: $C = f(x, y, z)$.

(b) Find $f(3000, 5000, 4000)$ and interpret it.

(c) What is the domain of f?

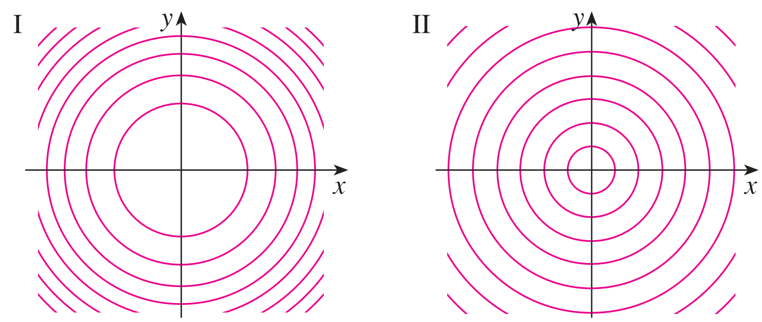
Solution: Replace this text with your solution. □

Exercise (14.1.33)**.** A contour map for a function f is shown. Use it to estimate the values of $f(-3, 3)$ and $f(3, -2)$. What can you say about the shape of the graph?



Solution: Replace this text with your solution. □

Exercise (14.1.36)**.** Two contour maps are shown. One is for a function f whose graph is a cone. The other is for a function g whose graph is a paraboloid. Which is which, and why?



Solution: Replace this text with your solution. □

Exercise (14.1.70)**.** Describe the level surfaces of the function $f(x, y, z) = x^2 - y^2 - z^2$.

Solution: Replace this text with your solution. □

Exercise (14.1.72)**.** Describe how the graph of g is obtained from the graph of f.

(a) $g(x, y) = f(x - 2, y)$

(b) $g(x, y) = f(x, y + 2)$

(c) $g(x, y) = f(x + 3, y - 4)$

Solution: Replace this text with your solution. □

Exercise (14.2.22)**.** Find

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2+y^2+z^2},$$

if it exists, or show that the limit does not exist.

Solution: Replace this text with your solution. □

Exercise (14.2.26)**.** Find $h(x,y) = g(f(x,y))$ and the set of points at which h is continuous if

$$g(t) = t + \ln t, \quad f(x,y) = \frac{1-xy}{1+x^2y^2}.$$

Solution: Replace this text with your solution. □

Exercise (14.2.38)**.** Determine the set of points at which the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+xy+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous.

Solution: Replace this text with your solution. □

Exercise (14.2.39)**.** Use polar coordinates to find the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}.$$

[If (r,θ) are polar coordinates of the point (x,y) with $r \geq 0$, note that $r \to 0^+$ as $(x,y) \to (0,0)$.]

Solution: Replace this text with your solution. □

Exercise (14.2.45)**.** Show that the function f given by $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$. [*Hint:* Consider $|\mathbf{x}-\mathbf{a}|^2 = (\mathbf{x}-\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a})$.]

Solution: Replace this text with your solution. □

Exercise (14.3.92)**.** One of Poiseuille's laws states that the resistance of blood flowing through an artery is

$$R = C\frac{L}{r^4}$$

where L and r are the length and radius of the artery and C is a positive constant determined by the viscosity of the blood. Calculate $\partial R/\partial L$ and $\partial R/\partial r$ and interpret them.

Solution: Replace this text with your solution. □

Exercise (14.3.96)**.** If a, b, c are the sides of a triangle and A, B, C are the opposite angles, find $\partial A/\partial a, \partial A/\partial b, \partial A/\partial c$ by implicit differentiation of the Law of Cosines.

Solution: Replace this text with your solution. □

Exercise (14.3.97)**.** You are told that there is a function f whose partial derivatives are $f_x(x, y) = x + 4y$ and $f_y(x, y) = 3x - y$. Should you believe it?

Solution: Replace this text with your solution. □

Exercise (14.3.101)**.** Use Clairaut's Theorem to show that if the third-order partial derivatives of f are continuous, then

$$f_{xyy} = f_{yxy} = f_{yyx}.$$

Solution: Replace this text with your solution. □

Exercise (14.3.104)**.** If $f(x, y) = \sqrt[3]{x^3 + y^3}$, find $f_x(0, 0)$.

Solution: Replace this text with your solution. □

Exercise (14.4.21)**.** Find the linear approximation of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at $(3, 2, 6)$ and use it to approximate the number $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$.

Solution: Replace this text with your solution. □

Exercise (14.4.34)**.** Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and metal in the sides is 0.05 cm thick.

Solution: Replace this text with your solution. □

Exercise (14.4.40)**.** A model for the surface area of a human body is given by $S = 0.1091w^{0.425}h^{0.725}$, where w is the weight (in pounds), h is the height (in inches), and S is measured in square feet. If the errors in measurement of w and h are at most 2%, use differentials to estimate the maximum percentage error in the calculated surface area.

Solution: Replace this text with your solution. □

Exercise (14.4.42)**.** Suppose you need to know an equation of the tangent plane to a surface S at the point $P(1, 2, 3)$. You don't have an equation for S but you know that the curves

$$\begin{aligned} \mathbf{r}_1(t) &= \langle 2 + 3t, 1 - t^2, 3 - 4t + t^2 \rangle \\ \mathbf{r}_2(t) &= \langle 1 + u^2, 2u^3 - 1, 2u + 1 \rangle \end{aligned}$$

both lie on S. Find an equation of the tangent plane at P.

Solution: Replace this text with your solution. □

Exercise (14.4.44)**.** Show that $f(x, y) = xy - 5y^2$ is differentiable by finding values of ε_1 and ε_2 that satisfy Definition 14.4.4.

Solution: Replace this text with your solution. □

Exercise (14.5.15)**.** Suppose f is a differentiable function of x and y, and $g(u, v) = f(e^u + \sin v, e^u + \cos v)$. Use the table of values to calculate $g_u(0, 0)$ and $g_v(0, 0)$.

	f	g	f_x	f_y
$(0, 0)$	3	6	4	8
$(1, 2)$	6	3	2	5

Solution: Replace this text with your solution. ☐

Exercise (14.5.44)**.** A sound with frequency f_s is produced by a source traveling along a line with speed v_s. If an observer is traveling with speed v_o along the same line from the opposite direction toward the source, then the frequency of the sound heard by observer is

$$f_o = \left(\frac{c + v_o}{c - v_s}\right) f_s$$

where c is the speed of sound, about 332 m/s. (This is the Doppler effect.) Suppose that, at a particular moment, you are in a train traveling at 34 m/s and accelerating at 1.2 m/s^2. A train is approaching you from the opposite direction on the other track at 40 m/s, accelerating at 1.4 m/s^2, and sounds its whistle, which has a frequency of 460 Hz. At that instant, what is the perceived frequency that you hear and how fast is it changing?

Solution: Replace this text with your solution. ☐

Exercise (14.5.53)**.** If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}.$$

Solution: Replace this text with your solution. ☐

Exercise (14.5.55)**.** A function f is called homogeneous of degree n if it satisfies the equation

$$f(tx, ty) = t^n f(x, y)$$

for all t, where n is a positive integer and f has continuous second-order partial derivatives.

(a) Verify that $f(x, y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3.

(b) Show that if f is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y).$$

[*Hint:* Use the Chain Rule to differentiate $f(tx, ty)$ with respect to t.]

Solution: Replace this text with your solution. □

Exercise (14.5.58)**.** Suppose that the equation $F(x, y, z) = 0$ implicitly defines each of the three variables x, y, and z as functions of the other two: $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$. If F is differentiable and F_x, F_y, and F_z are all nonzero, show that

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -1.$$

Solution: Replace this text with your solution. □

Exercise (14.6.30)**.** Near a buoy, the depth of a lake at the point with coordinates (x, y) is $z = 200 + 0.02x^2 - 0.001y^3$, where x, y, and z are measured in meters. A fisherman in a small boat starts at the point $(80, 60)$ and moves toward the buoy, which is located at $(0, 0)$. Is the water under the boat getting deeper or shallower when he departs? Explain.

Exercise (14.6.33)**.** Suppose that over a certain region of space the electrical potential V is given by $V(x, y, z) = 5x^2 - 3xy + xyz$.

(a) Find the rate of change of the potential at $P(3, 4, 5)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

(b) In which direction does V change most rapidly at P?

(c) Find the maximum rate of increase at P.

Exercise (14.6.40)**.** The <u>second directional derivative</u> of $f(x, y)$ is

$$D_{\mathbf{u}}^2 f(x, y) = D_{\mathbf{u}}[D_{\mathbf{u}} f(x, y)].$$

(a) If $\mathbf{u} = \langle a, b\rangle$ is a unit vector and f has continuous second partial derivatives, show that

$$D_{\mathbf{u}}^2 f = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2.$$

(b) Find the second directional derivative of $f(x, y) = xe^{2y}$ in the direction of $\mathbf{v} = \langle 4, 6\rangle$.

Solution: Replace this text with your solution. □

Exercise (14.6.61)**.** Show that the sum of the x-, y-, and z-intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

Solution: Replace this text with your solution. □

Exercise (14.6.70)**.** Show that if $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = \langle x_0, y_0\rangle$, then

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0)\cdot(\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0.$$

[*Hint:* Use Definition 14.4.4 directly.]

Solution: Replace this text with your solution. □

Exercise (14.7.40)**.** If a function of one variable is continuous on an interval and has only one critical number, then a local maximum has to be an absolute maximum. But this is not true for functions of two variables. Show that the function

$$f(x, y) = 3xe^y - x^3 - e^{3y}$$

has exactly one critical point, and that f has a local maximum there that is not an absolute maximum. Then use a computer to produce a graph with a carefully chosen domain and viewpoint to see how this is possible.

Solution: Replace this text with your solution. □

Exercise (14.7.52)**.** The base of an aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials.

Solution: Replace this text with your solution. □

Exercise (14.7.55)**.** If the length of the diagonal of a rectangular box must be L, what is the largest possible volume?

Solution: Replace this text with your solution. □

Exercise (14.7.56)**.** A model for the yield Y of an agricultural crop as a function of the nitrogen level N and phosphorus level P in the soil (measured in appropriate units) is

$$Y(N, P) = kNPe^{-N-P}$$

where k is a positive constant. What levels of nitrogen and phosphorus result in the best yield?

Solution: Replace this text with your solution. □

Exercise (14.7.58)**.** Three alleles (alternative versions of a gene) A, B, and O determine the four blood types A (AA or AO), B (BB or BO), O (OO), and AB. The Hardy-Weinberg Law states that the proportion of individuals in a population who carry two different alleles is

$$P = 2pq + 2pr + 2rq$$

where p, q, and r are the proportions of A, B, and O in the population. Use the fact that $p + q + r = 1$ to show that P is at most $\frac{2}{3}$.

Solution: Replace this text with your solution. □

Exercise (14.8.15)**.** The method of Lagrange multipliers assumes that the extreme values exist, but that is not always the case. Show that the problem of finding the minimum value of $f(x, y) = x^2 + y^2$ subject to the constraint $xy = 1$ can be solved using Lagrange multipliers, but f does not have a maximum value with that constraint.

Solution: Replace this text with your solution. □

Exercise (14.8.29)**.** Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.

Solution: Replace this text with your solution. □

Exercise (14.8.30)**.** Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter p is equilateral.

Hint: Use Heron's formula for the area:

$$A = \sqrt{s(s-x)(s-y)(s-z)}$$

where $s = p/2$ and x, y, z are the lengths of the sides.

Solution: Replace this text with your solution. □

Exercise (14.8.49)**.** (a) Find the maximum value of

$$f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$$

given that $x_1, x_2, \ldots, x_n$ are positive numbers and $x_1 + x_2 + \cdots + x_n = c$, where c is a constant.

(b) Deduce from part (a) that if $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

This inequality says that the geometric mean of n numbers is no larger than the arithmetic mean of the numbers. Under what circumstances are these two means equal?

Solution: Replace this text with your solution. □

Exercise (14.8.50)**.** (a) Maximize $\sum_{i=1}^{n} x_i y_i$ subject to the constraints $\sum_{i=1}^{n} x_i^2 = 1$ and $\sum_{i=1}^{n} y_i^2 = 1$.

(b) Put
$$x_i = \frac{a_i}{\sqrt{\sum a_j^2}} \quad \text{and} \quad y_i = \frac{b_i}{\sqrt{\sum b_j^2}}$$
to show that
$$\sum a_i b_i \leq \sqrt{\sum a_j^2}\sqrt{\sum b_j^2}$$
for any numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$. This inequality is known as the Cauchy-Schwarz Inequality.

Solution: Replace this text with your solution. $\square$