Multivariable Calculus Homework #4

Replace this text with your name

Due: Replace this text with a due date

Exercise (15.1.6)**.** A 20-ft-by-30-ft swimming pool is filled with water. The depth is measured at 5-ft intervals, starting at one corner of the pool, and the values are recorded in the table. Estimate the volume of water in the pool.

	0	5	10	15	20	25	30
0	2	3	4	6	7	8	8
5	2	3	4	7	8	10	8
10	2	4	6	8	10	12	10
15	2	3	4	5	6	8	7
20	2	2	2	2	3	4	4

Solution: Replace this text with your solution. □

Exercise (15.1.11)**.** Evaluate the double integral

$$\iint_R (4-2y)\,dy, \quad R=[0,1]\times[0,1]$$

by first identifying it as the volume of a solid.

Solution: Replace this text with your solution. □

Exercise (15.1.43)**.** Find the volume of the solid enclosed by the paraboloid $z=2+x^2+(y-2)^2$ and the planes $z=1$, $x=1$, $x=-1$, $y=0$, and $y=4$.

Solution: Replace this text with your solution. □

Exercise (15.1.49)**.** Use symmetry to evaluate the double integral

$$\iint_R \frac{xy}{1+x^4}\,dA, \quad R=\{(x,y)\mid -1\le x\le 1, 0\le y\le 1\}.$$

Solution: Replace this text with your solution. □

Exercise (15.1.51)**.** Use a computer algebra system to compute the iterated integrals

$$\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx \qquad \text{and} \qquad \int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy.$$

Do the answers contradict Fubini's Theorem? Explain what is happening.

Solution: Replace this text with your solution. □

Exercise (15.2.37)**.** Find the volume of the solid under the plane $z = 3$, above the plane $z = y$, and between the parabolic cylinders $y = x^2$ and $y = 1 - x^2$, by subtracting two volumes.

Solution: Replace this text with your solution. □

Exercise (15.2.56)**.** Evaluate the integral

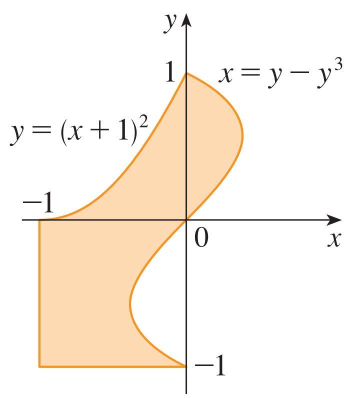
$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$$

by reversing the order of integration.

Solution: Replace this text with your solution. □

Exercise (15.2.58)**.** Express D as a union of regions of type I or type II and evaluate the integral

$$\iint_D y\, dA.$$



Solution: Replace this text with your solution. □

Exercise (15.2.63)**.** Prove that if $m \leq f(x,y) \leq M$ for all x, y in D, then

$$mA(D) \leq \iint_D f(x,y)\, dA \leq MA(D).$$

Solution: Replace this text with your solution. □

Exercise (15.2.69)**.** Use geometry or symmetry, or both, to evaluate the double integral

$$\iint_D \left(ax^3 + by^3 + \sqrt{a^2 - x^2}\right) dA, \quad D = [-a,a] \times [-b,b].$$

Solution: Replace this text with your solution. □

Exercise (15.3.39)**.** Use polar coordinates to combine the sum

$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_{1}^{\sqrt{2}}\int_{0}^{x} xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\,dy\,dx$$

into one double integral. Then evaluate the double integral.

Solution: Replace this text with your solution. □

Exercise (15.3.40)**.** (a) We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$\begin{aligned} I &= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dy\,dx \\ &= \lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}\,dA, \end{aligned}$$

where D_a is the disk with radius a and center the origin. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA = \pi.$$

(b) An equivalent definition of the improper integral in part (a) is

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \lim_{a\to\infty}\iint_{S_a} e^{-(x^2+y^2)}\,dA,$$

where S_a is the square with vertices $(\pm a, \pm a)$. Use this to show that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \pi.$$

(c) Deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

(d) By making the change of variable $t = \sqrt{2}x$, show that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}.$$

(This is a fundamental result for probability and statistics.)

Solution: Replace this text with your solution. □

Exercise (15.4.15)**.** Find the center of mass of a lamina in the shape of an isosceles right triangle with equal sides of length a if the density at any point is proportional to the square of the distance from the vertex opposite the hypotenuse.

Solution: Replace this text with your solution. □

Exercise (15.4.20)**.** Consider a square fan blade with sides of length 2 and the lower left corner placed at the origin. If the density of the blade is $\rho(x, y) = 1 + 0.1x$, is it more difficult to rotate the blade about the x-axis or the y-axis?

Solution: Replace this text with your solution. □

Exercise (15.4.30)**.**

(a) A lamp has two bulbs, each of a type with average lifetime 1000 hours. Assuming that we can model the probability of failure of a bulb by an exponential density function with mean $\mu = 1000$, find the probability that both of the lamp's bulbs fail within 1000 hours.

(b) Another lamp has just one bulb of the same type as in part (a). If one bulb burns out and is replaced by a bulb of the same type, find the probability that the two bulbs fail within a total of 1000 hours.

Solution: Replace this text with your solution. □

Exercise (15.4.32)**.** Xavier and Yolanda both have classes that end at noon and they agree to meet every day after class. They arrive at the coffee shop independently. Xavier's arrival time is X and Yolanda's arrival time is Y, where X and Y are measured in minutes after noon. The individual density functions are

$$f_1(x) = \begin{cases} e^{-x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases} \qquad f_2(y) = \begin{cases} \frac{1}{50}y, & \text{if } 0 \leq y \leq 10, \\ 0, & \text{otherwise.} \end{cases}$$

(Xavier arrives sometime after noon and is more likely to arrive promptly than late. Yolanda always arrives by 12:10 PM and is more likely to arrive late than promptly.) After Yolanda arrives, she'll wait for up to half an hour for Xavier, but he won't wait for her. Find the probability that they meet.

Solution: Replace this text with your solution. □

Exercise (15.5.7)**.** Find the area of the part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution: Replace this text with your solution. □

Exercise (15.5.14)**.** Find the area of the surface $z = \cos(x^2 + y^2)$ that lies inside the cylinder $x^2 + y^2 = 1$ correct to four decimal places by expressing the area in terms of a single integral and using your calculator to estimate the integral.

Solution: Replace this text with your solution. □

Exercise (15.5.19)**.** Use a calculator to find the exact area of the surface

$$z = 1 + x + y + x^2 \qquad -2 \leq x \leq 1 \quad -1 \leq y \leq 1.$$

Illustrate by graphing the surface.

Exercise (15.5.21)**.** Show that the area of the part of the plane $z = ax + by + c$ that projects onto a region D in the xy-plane with area $A(D)$ is $\sqrt{a^2 + b^2 + 1}\, A(D)$.

Solution: Replace this text with your solution. □

Exercise (15.5.23)**.** Find the area of the finite part of the paraboloid $y = x^2 + z^2$ cut off by the plane $y = 25$. [*Hint:* Project the surface onto the xz-plane.]

Solution: Replace this text with your solution. □

Exercise (15.6.24)**.** (a) In the Midpoint Rule for triple integrals we use a triple Riemann sum to approximate a triple integral over a box B, where $f(x, y, z)$ is evaluated at the center $(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ of the box B_{ijk}. Use the Midpoint Rule to estimate $\iiint_B \sqrt{x^2 + y^2 + z^2}\, dV$, where B is the cube defined by $0 \leq x \leq 4$, $0 \leq y \leq 4$, $0 \leq z \leq 4$. Divide B into eight cubes of equal size.

(b) Use a computer algebra system to approximate the integral in part (a) correct to the nearest integer. Compare with the answer to part (a).

Solution: Replace this text with your solution. □

Exercise (15.6.36)**.** Write five other iterated integrals that are equal to the iterated integral

$$\int_0^1 \int_y^1 \int_0^z f(x, y, z)\, dx\, dz\, dy.$$

Solution: Replace this text with your solution. □

Exercise (15.6.38)**.** Evaluate the triple integral $\iiint_B (z^3 + \sin y + 3)\, dV$, where B is the unit ball $x^2 + y^2 + z^2 \leq 1$, using only geometric interpretation and symmetry.

Solution: Replace this text with your solution. □

Exercise (15.6.47)**.** Set up, but do not evaluate, integral expressions for (a) the mass, (b) the center of mass, and (c) the moment of inertia about the z-axis for the hemisphere $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$; $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

Solution: Replace this text with your solution. □

Exercise (15.6.53)**.** The average value of a function $f(x, y, z)$ over a solid region E is defined to be

$$f_{\text{ave}} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV$$

where $V(E)$ is the volume of E. For instance, if ρ is a density function then ρ_{ave} is the average density of E.
Find the average value of the function $f(x, y, z) = xyz$ over the cube with side length L that lies in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

Solution: Replace this text with your solution. □

Exercise (15.7.13)**.** A cylindrical shell is 20 cm long, with inner radius 6 cm and outer radius 7 cm. Write inequalities that describe the shell in an appropriate coordinate system. Explain how you have positioned the coordinate system with respect to the shell.

Solution: Replace this text with your solution. □

Exercise (15.7.22)**.** Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

Solution: Replace this text with your solution. □

Exercise (15.7.30)**.** Evaluate the integral

$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} \sqrt{x^2 + y^2}\, dz\, dy\, dx$$

by changing to cylindrical coordinates.

Solution: Replace this text with your solution. □

Exercise (15.7.31)**.** When studying the formation of mountain ranges, geologists estimate the amount of work required to lift a mountain from sea level. Consider a mountain that is essentially in the shape of a right circular cone. Suppose that the weight density of the material in the vicinity of a point P is $g(P)$ and the height is $h(P)$.

(a) Find a definite integral that represents the total work done in forming the mountain.

(b) Assume that Mount Fuji in Japan is in the shape of a right circular cone with radius 62,000 ft, height 12,400 ft, and density a constant 200 lb/ft^3. How much work was done in forming Mount Fuji if the land was initially at sea level?

Solution: Replace this text with your solution. □

Exercise (15.7.36)**.** Find the volume of the smaller wedge cut from a sphere of radius a by two planes that intersect along a diameter at an angle of $\pi/6$.

Solution: Replace this text with your solution. □

Exercise (15.7.41)**.** Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\, dz\, dy\, dx$$

by changing to spherical coordinates.

Solution: Replace this text with your solution. □

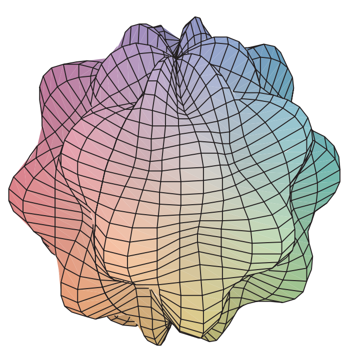
Exercise (15.8.47)**.** A model for the density δ of the earth's atmosphere near its surface is

$$\delta = 619.09 - 0.000097\rho$$

where ρ (the distance from the center of the earth) is measured in meters and δ is measured in kilograms per cubic meter. If we take the surface of the earth to be a sphere with radius 6370 km, then this model is a reasonable one for $6.370 \times 10^6 \leq \rho \leq 6.375 \times 10^6$. Use this model to estimate the mass of the atmosphere between the ground and an altitude of 5 km.

Solution: Replace this text with your solution. □

Exercise (15.8.47)**.** The surfaces $\rho = 1 + \frac{1}{5}\sin m\theta \sin n\phi$ have been used as models for tumors. The "bumpy sphere" with $m = 6$ and $n = 5$ is shown. Use a computer algebra system to find the volume it encloses.



Solution: Replace this text with your solution. □

Exercise (15.7.48)**.** Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz = 2\pi.$$

(The improper triple integral is defined as the limit of a triple integral over a solid sphere as the radius of the sphere increases indefinitely.)

Solution: Replace this text with your solution. □

Exercise (15.8.14)**.** Find equations for a transformation T that maps a rectangular region S in the uv-plane onto the region R in the xy-plane bounded by the hyperbolas $y = 1/x$, $y = 4/x$ and the lines $y = x$, $y = 4x$ in the first quadrant, where the sides of S are parallel to the u- and v-axes.

Solution: Replace this text with your solution. □

Exercise (15.8.18)**.** Use the transformation $x = \sqrt{2}u - \sqrt{2/3}v$, $y = \sqrt{2}u + \sqrt{2/3}v$ to evaluate the integral $\iint_R (x^2 - xy + y^2)\, dA$, where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$.

Solution: Replace this text with your solution. □

Exercise (15.8.22)**.** An important problem in thermodynamics is to find the work done by an ideal Carnot engine. A cycle consists of alternating expansion and compression of gas in a piston. The work done by the engine is equal to the area of the region R enclosed by two isothermal curves $xy = a$, $xy = b$ and two adiabatic curves $xy^{1.4} = c$, $xy^{1.4} = d$, where $0 < a < b$ and $0 < c < d$. Compute the work done by determining the area of R.

Solution: Replace this text with your solution. □

Exercise (15.8.27)**.** Evaluate the integral $\iint_R e^{x+y}\, dA$, where R is given by the inequality $|x| + |y| \leq 1$.

Solution: Replace this text with your solution. □

Exercise (15.8.28)**.** Let f be continuous on $[0, 1]$ and let R be the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Show that

$$\iint_R f(x + y)\, dA = \int_0^1 u f(u)\, du.$$

Solution: Replace this text with your solution. □