Multivariable Calculus Notes

Manhattan High School

Teacher: Asher Roberts
Principal: Marta Barnett

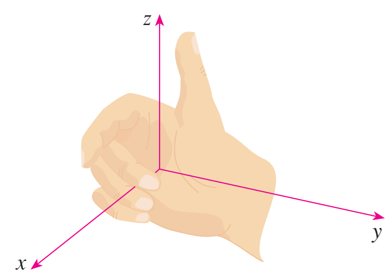
2019-2020

Contents

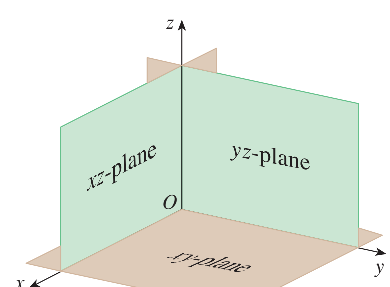
Chapter 12

Vectors and the Geometry of Space

12.1 Three-Dimensional Coordinate Systems



Definition 12.1.1. The coordinate axes are three directed lines through the origin that are perpendicular to each other and labeled the x-axis, y-axis, and z-axis. The direction of the z-axis is determined by the right-hand rule as illustrated in the figure.



Definition 12.1.2. The three coordinate axes determine the three coordinate planes illustrated in the figure. These three coordinate planes divide space into eight parts, called octants. The first octant, in the foreground of the figure, is determined by the positive axes.

Definition 12.1.3. We represent a point P in space by the ordered triple (a, b, c) where a is the distance from the yz-plane to P, b is the distance from the xz-plane to P, and c is the distance from the xy-plane to P. We call a, b, and c the coordinates of P. The points $(a, b, 0)$, $(0, b, c)$, and $(a, 0, c)$ are called the projections of P onto the xy-plane, yz-plane, and xz-plane, respectively.

Definition 12.1.4. The Cartesian product $\mathbb{R}\times\mathbb{R}\times\mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. It is called a three-dimensional rectangular coordinate system.

Example 1. What surfaces in $\mathbb{R}^3$ are represented by the following equations?

(a) $z = 3$

(b) $y = 5$

Example 2. (a) Which points (x, y, z) satisfy the equations

$$x^2 + y^2 = 1 \qquad \text{and} \qquad z = 3?$$

(b) What does the equation $x^2 + y^2 = 1$ represent as a surface in $\mathbb{R}^3$?

Example 3. Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $y = x$.

Theorem 12.1.1 (Distance Formula in Three Dimensions)**.** *The distance* $|P_1P_2|$ *between the points* $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$ *is*

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 4. Find the distance from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$.

Example 5. Find an equation of a sphere with radius r and center $C(h, k, l)$.

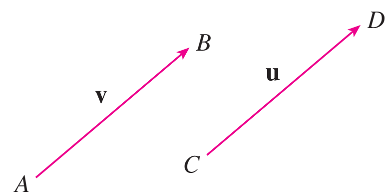
Example 6. Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

Example 7. What region in $\mathbb{R}^3$ is represented by the following inequalities?

$$1 \leq x^2 + y^2 + z^2 \leq 4 \qquad z \leq 0.$$

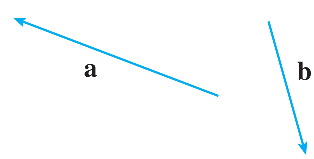
12.2 Vectors

Definition 12.2.1. A vector is a quantity that has both magnitude and direction, denoted $\mathbf{v}$ or $\vec{v}$. For a particle that moves along a line segment from point A to point B, the corresponding displacement vector, shown in the figure, has initial point A and terminal point B and we indicate this by writing $\mathbf{v} = \overrightarrow{AB}$. Because the vector $\mathbf{u} = \overrightarrow{CD}$ has the same length and the same direction as $\mathbf{v}$, even though it is in a different position, we say that $\mathbf{u}$ and $\mathbf{v}$ are equivalent (or equal) and we write $\mathbf{u} = \mathbf{v}$. The zero vector, denoted by $\mathbf{0}$ has length 0.



Definition 12.2.2 (Vector Addition)**.** If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the sum $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

Example 1. Draw the sum of the vectors $\mathbf{a}$ and $\mathbf{b}$ shown in the figure.



Definition 12.2.3 (Scalar Multiplication)**.** If c is a scalar and $\mathbf{v}$ is a vector, then the scalar multiple $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

Definition 12.2.4. Two nonzero vectors are parallel if they are scalar multiples of one another. In particular, the vector $-\mathbf{v} = (-1)\mathbf{v}$, called the negative of $\mathbf{v}$, has the same length as $\mathbf{v}$ but points in the opposite direction. By the difference $\mathbf{u} - \mathbf{v}$ of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

Example 2. If **a** and **b** are the vectors shown in the figure, draw $\mathbf{a} - 2\mathbf{b}$.

Definition 12.2.5. If we place the initial point of a vector **a** at the origin of a rectangular coordinate system, then the terminal point of **a** has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3). These coordinates are called the <u>components</u> of **a** and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \qquad \text{or} \qquad \mathbf{a} = \langle a_1, a_2, a_3 \rangle.$$

The <u>representation</u> of a vector from the origin to a point is called the <u>position vector</u> of the point.

Theorem 12.2.1. *Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector* **a** *with representation $\overrightarrow{AB}$ is*

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

Proof. The vector $\mathbf{a} = \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ is the position vector of the point $P(a_1, a_2, a_3)$. If $\overrightarrow{AB}$ is another representation of **a**, where the initial point is $A(x_1, y_1, z_1)$ and the terminal point is $B(x_2, y_2, z_2)$, then we must have $x_1 + a_1 = x_2$, $y_1 + a_2 = y_2$, and $z_1 + a_3 = z_2$. Therefore, $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, and $a_3 = z_2 - z_1$. □

Example 3. Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

Definition 12.2.6. The <u>magnitude</u> or <u>length</u> of the vector $\mathbf{v}$ is the length of any of its representations and is denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$.

Theorem 12.2.2. *The length of the two-dimensional vector* $\mathbf{a} = \langle a_1, a_2 \rangle$ *is*

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}.$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ *is*

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Theorem 12.2.3. *If* $\mathbf{a} = \langle a_1, a_2 \rangle$ *and* $\mathbf{b} = \langle b_1, b_2 \rangle$*, then*

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

and

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

for a scalar c*. Similarly, for three-dimensional vectors,*

$$\begin{aligned} \langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle &= \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle \\ \langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle &= \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle \\ c\langle a_1, a_2, a_3 \rangle &= \langle ca_1, ca_2, ca_3 \rangle. \end{aligned}$$

Example 4. If $\mathbf{a} = \langle 4, 0, 3 \rangle$ and $\mathbf{b} = \langle -2, 1, 5 \rangle$, find $|\mathbf{a}|$ and the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $3\mathbf{b}$, and $2\mathbf{a} + 5\mathbf{b}$.

Definition 12.2.7. We denote by V_2 the set of all two-dimensional vectors and by V_3 the set of all three-dimensional vectors. More generally, we denote by V_n the set of all n-dimensional vectors. An n-dimensional vector is an ordered n-tuple:

$$\mathbf{a} = \langle a_1, a_2, \dots, a_n \rangle$$

where $a_1, a_2, \dots, a_n$ are real numbers that are called the components of $\mathbf{a}$. Addition and scalar multiplication are defined in terms of components just as for the cases $n = 2$ and $n = 3$.

Theorem 12.2.4 (Properties of Vectors)**.** *If* $\mathbf{a}$*,* $\mathbf{b}$*, and* $\mathbf{c}$ *are vectors in* V_n *and* c *and* d *are scalars, then*

tsk$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

tsk$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

tsk$\mathbf{a} + \mathbf{0} = \mathbf{a}$

tsk$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

tsk$c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

tsk$(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$

tsk$(cd)\mathbf{a} = c(d\mathbf{a})$

tsk$1\mathbf{a} = \mathbf{a}$

Definition 12.2.8. The vectors

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

are called the standard basis vectors. They have length 1 and point in the directions of the positive x-, y-, and z-axes. Similarly, in two dimensions we define $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

Theorem 12.2.5. *Any vector in* V_3 *can be expressed in terms of* $\mathbf{i}$*,* $\mathbf{j}$*, and* $\mathbf{k}$*. Similarly, any vector in* V_2 *can be expressed in terms of* $\mathbf{i}$ *and* $\mathbf{j}$*.*

Proof. If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\begin{aligned}\mathbf{a} = \langle a_1, a_2, a_3 \rangle &= \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle \\ &= a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}.\end{aligned}$$

Similarly, in two dimensions, we can write

$$\mathbf{a} = \langle a_1, a_2 \rangle = a_1 \mathbf{i} + a_2 \mathbf{j}.$$

□

Example 5. If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 7\mathbf{k}$, express the vector $2\mathbf{a} + 3\mathbf{b}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Definition 12.2.9. A unit vector is a vector whose length is 1. For instance, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are all unit vectors.

Theorem 12.2.6. *In general, if* $\mathbf{a} \neq \mathbf{0}$*, then the unit vector that has the same direction as* $\mathbf{a}$ *is*

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

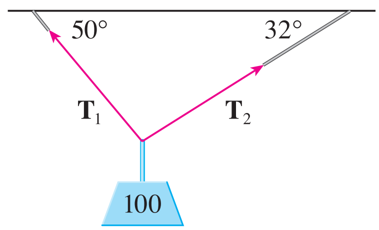
Proof. Let $c = 1/|\mathbf{a}|$. Then $\mathbf{u} = c\mathbf{a}$ and c is a positive scalar, so $\mathbf{u}$ has the same direction as $\mathbf{a}$. Also

$$|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1.$$ □

Example 6. Find the unit vector in the direction of the vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

Definition 12.2.10. A force is represented by a vector because it has both a magnitude (measured in pounds or newtons) and a direction. If several forces are acting on an object, the resultant force experienced by the object is the vector sum of these forces.

Example 7. A 100-lb weight hangs from two wires as shown in the figure. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and the magnitudes of the tensions.



12.3 The Dot Product

Definition 12.3.1. If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the <u>dot product</u> of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and similarly

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

for two-dimensional vectors.

Example 1. Compute the following dot products:

(a) $\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle$

(b) $\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle$

(c) $(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k})$

Theorem 12.3.1 (Properties of the Dot Product)**.** *If* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are vectors in* V_3 *and* c *is a scalar, then*

tsk$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

tsk$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

tsk$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

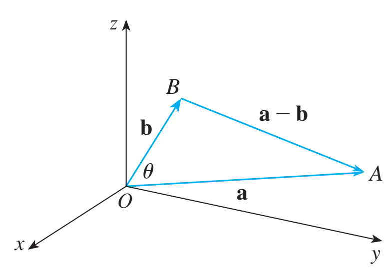
tsk$(c\mathbf{a}) \cdot (\mathbf{b}) = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

tsk$\mathbf{0} \cdot \mathbf{a} = 0$

Theorem 12.3.2. *If* θ *is the angle between the vectors* $\mathbf{a}$ *and* $\mathbf{b}$, *then*

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta.$$

Proof. If we apply the Law of Cosines to triangle OAB in the figure, we get



$$\begin{aligned}
|AB|^2 &= |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta \\
|\mathbf{a} - \mathbf{b}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta \\
(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta \\
\mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta \\
|\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta \\
-2\mathbf{a} \cdot \mathbf{b} &= -2|\mathbf{a}||\mathbf{b}|\cos\theta \\
\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos\theta
\end{aligned}$$

□

Example 2. If the vectors $\mathbf{a}$ and $\mathbf{b}$ have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

Corollary 12.3.1. *If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then*

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

Example 3. Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -3, 2 \rangle$.

Definition 12.3.2. Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are called perpendicular or orthogonal if the angle between them is $\theta = \pi/2$. The zero vector $\mathbf{0}$ is considered to be perpendicular to all vectors.

Theorem 12.3.3. *Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.*

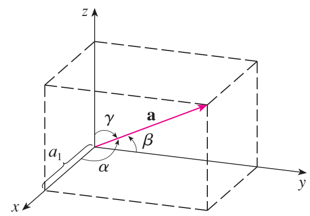
Proof. If $\theta = \pi/2$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/2) = 0.$$

Conversely, if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos\theta = 0$, so $\theta = \pi/2$. □

Example 4. Show that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

Definition 12.3.3. The direction angles of a nonzero vector $\mathbf{a}$ are the angles α, β, and γ (in the interval $[0, \pi]$) that $\mathbf{a}$ makes with the positive x-, y-, and z-axes, respectively. (See the figure.) The cosines of these direction angles, $\cos\alpha$, $\cos\beta$, $\cos\gamma$, are called the direction cosines of the vector $\mathbf{a}$.



Theorem 12.3.4. *The direction cosines of a vector* $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ *are the components of the unit vector in the direction of* $\mathbf{a}$*, i.e.,*

$$\frac{1}{|\mathbf{a}|}\mathbf{a} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle.$$

Proof. By Corollary 12.3.1,

$$\cos\alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}.$$

Similarly,

$$\cos\beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|}.$$

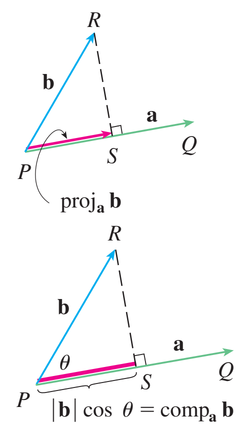
Therefore,

$$\begin{aligned} \mathbf{a} &= \langle a_1, a_2, a_3 \rangle \\ \mathbf{a} &= \langle |\mathbf{a}|\cos\alpha, |\mathbf{a}|\cos\beta, |\mathbf{a}|\cos\gamma \rangle \\ \mathbf{a} &= |\mathbf{a}|\langle \cos\alpha, \cos\beta, \cos\gamma \rangle \\ \frac{1}{|\mathbf{a}|}\mathbf{a} &= \langle \cos\alpha, \cos\beta, \cos\gamma \rangle. \end{aligned}$$

□

Example 5. Find the direction angles of the vector $\mathbf{a} = \langle 1, 2, 3 \rangle$.

Definition 12.3.4. If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ and is denoted by $\text{proj}_{\mathbf{a}}\,\mathbf{b}$. (See the figure.)
The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ (also called the component of $\mathbf{b}$ along $\mathbf{a}$) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}|\cos\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. (See the figure.) This is denoted by $\text{comp}_{\mathbf{a}}\,\mathbf{b}$.



Theorem 12.3.5. *The scalar projection of* $\mathbf{b}$ *onto* $\mathbf{a}$ *is*

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}.$$

The vector projection of $\mathbf{b}$ *onto* $\mathbf{a}$ *is*

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}.$$

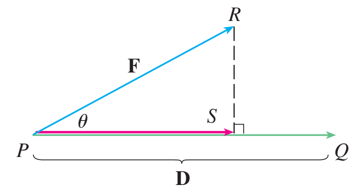
Proof. By Theorem 12.3.2,

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$
$$\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = |\mathbf{b}|\cos\theta,$$

which gives us the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$. Multiplying by the unit vector gives us the vector projection in the direction of $\mathbf{a}$. □

Example 6. Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2\rangle$ onto $\mathbf{a} = \langle -2, 3, 1\rangle$.

Definition 12.3.5. Suppose that the constant force in moving an object from P to Q is $\mathbf{F} = \overrightarrow{PR}$, as in the figure. Then the displacement vector is $\mathbf{D} = \overrightarrow{PQ}$ and the work done by this force is defined to be the product of the component of the force along $\mathbf{D}$ and the distance moved:



$$W = \big(|\mathbf{F}| \cos\theta\big)\, |\mathbf{D}|.$$

Theorem 12.3.6. *The work done by a constant force* $\mathbf{F}$ *is the dot product* $\mathbf{F} \cdot \mathbf{D}$*, where* $\mathbf{D}$ *is the displacement vector.*

Proof. By Theorem 12.3.2,

$$W = |\mathbf{F}||\mathbf{D}| \cos\theta = \mathbf{F} \cdot \mathbf{D}. \qquad \square$$

Example 7. A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal path. Find the work done by the force.

Example 8. A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.

12.4 The Cross Product

Definition 12.4.1. If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle.$$

Definition 12.4.2. A determinant of order 2 is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

A determinant of order 3 is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Theorem 12.4.1. *The cross product of the vectors* $\mathbf{a} = a_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ *and* $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ *is*

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}.$$

Example 1. If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, find $\mathbf{a} \times \mathbf{b}$.

Example 2. Show that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$ in V_3.

Theorem 12.4.2. *The vector* $\mathbf{a} \times \mathbf{b}$ *is orthogonal to both* $\mathbf{a}$ *and* $\mathbf{b}$.

Proof.

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1) \\ &= a_1a_2b_3 - a_1b_2a_3 - a_1a_2b_3 + b_1a_2a_3 + a_1b_2a_3 - b_1a_2a_3 \\ &= 0.\end{aligned}$$

Similarly, $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$. □

Theorem 12.4.3. *If* θ *is the angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *(so* $0 \le \theta \le \pi$*), then*

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta.$$

Proof.

$$\begin{aligned}|\mathbf{a} \times \mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\ &= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 \\ &\qquad + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2 \cos^2\theta \\ &= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta) \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2\theta.\end{aligned}$$

$\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \geq 0$ when $0 \leq \theta \leq \pi$, so

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta. \qquad \square$$

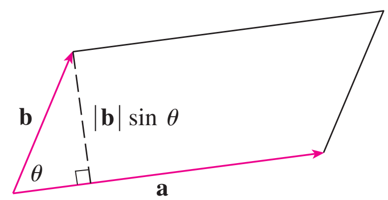
Corollary 12.4.1. *Two nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel if and only if*

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}.$$

Proof. Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\theta = 0$ or π. In either case $\sin\theta = 0$, so $|\mathbf{a} \times \mathbf{b}| = 0$ and therefore $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. $\square$

Corollary 12.4.2. *The length of the cross product* $\mathbf{a} \times \mathbf{b}$ *is equal to the area of the parallelogram determined by* $\mathbf{a}$ *and* $\mathbf{b}$*.*

Proof. The geometric interpretation of Theorem 12.4.3. can be seen by looking at the figure. If $\mathbf{a}$ and $\mathbf{b}$ are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $|\mathbf{b}|\sin\theta$, and area



$$A = |\mathbf{a}|(|\mathbf{b}|\sin\theta) = |\mathbf{a} \times \mathbf{b}|. \qquad \square$$

Example 3. Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Example 4. Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Theorem 12.4.4. *If* $\mathbf{a}$*,* $\mathbf{b}$*, and* $\mathbf{c}$ *are vectors and* c *is a scalar, then*

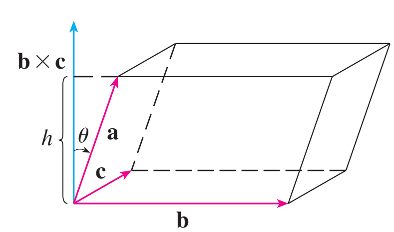
1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Theorem 12.4.5. *The volume of the parallelepiped determined by the vectors* $\mathbf{a}$*,* $\mathbf{b}$*, and* $\mathbf{c}$ *is the magnitude of their* *scalar triple product:*

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \left\| \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right\|.$$

If the volume of the parallelepiped determined by $\mathbf{a}$*,* $\mathbf{b}$*, and* $\mathbf{c}$ *is 0, then the vectors must lie in the same plane; that is, they are* *coplanar.*

Proof. The geometric interpretation of the scalar triple product can be seen by looking at the figure. The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$. If θ is the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. Therefore the volume of the parallelepiped is



$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|. \qquad \square$$

Example 5. Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 0, -9, 18 \rangle$ are coplanar.

Definition 12.4.3. If $\mathbf{F}$ is a force acting on a rigid body at a point given by a position vector $\mathbf{r}$ then the torque $\boldsymbol{\tau}$ (relative to the origin) is defined to be the cross product of the position and force vectors

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

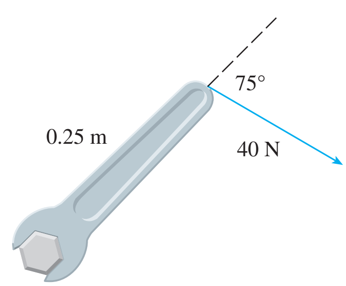
and measures the tendency of the body to rotate about the origin.

Theorem 12.4.6. *The magnitude of the torque vector is*

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin\theta$$

where θ is the angle between the position and force vectors.

Example 6. A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in the figure. Find the magnitude of the torque about the center of the bolt.

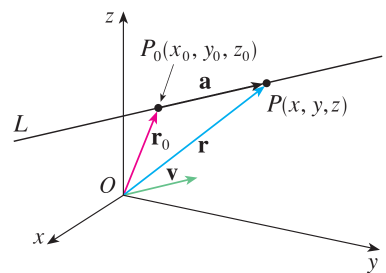


12.5 Equations of Lines and Planes

Theorem 12.5.1. *The vector equation of a line through the point* (x_0, y_0, z_0) *is*

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r}_0$ *is the position vector of* (x_0, y_0, z_0), $\mathbf{v}$ *is a vector parallel to the line, and* t *is a scalar.*
Parametric equations for a line through the point (x_0, y_0, z_0) *and parallel to the direction vector* $\langle a, b, c\rangle$ *are*



$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct.$$

Example 1. (a) Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

(b) Find two other points on the line.

Definition 12.5.1. In general, if a vector $\mathbf{v} = \langle a, b, c\rangle$ is used to describe the direction of a line L, then the numbers a, b, and c are called the direction numbers of L. The equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

obtained by eliminating the parameter t are called symmetric equations of L.

Example 2. (a) Find parametric equations and symmetric equations of the line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.

(b) At what point does this line intersect the xy-plane?

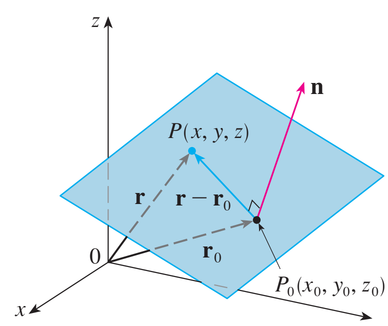
Theorem 12.5.2. *The line segment from* $\mathbf{r}_0$ *to* $\mathbf{r}_1$ *is given by the vector equation*

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \le t \le 1.$$

Example 3. Show that the lines L_1 and L_2 with parametric equations

$$L_1: \quad x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$
$$L_2: \quad x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are skew lines; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).



Definition 12.5.2. Either

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

or

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

is called a vector equation of a plane through point (x_0, y_0, z_0) where $\mathbf{r}_0$ is the position vector of (x_0, y_0, z_0), $\mathbf{r}$ is the vector equation of the line through (x_0, y_0, z_0), and $\mathbf{n}$ is the vector through (x_0, y_0, z_0) orthogonal to the plane, called a normal vector.
A scalar equation of the plane through point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Example 4. Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$. Find the intercepts and sketch the plane.

Theorem 12.5.3. *The equation of a plane can be rewritten as the* <u>*linear equation*</u>

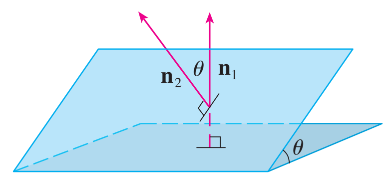
$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$.

Example 5. Find an equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

Example 6. Find the point at which the line with parametric equations $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

Definition 12.5.3. Two planes are parallel if their normal vectors are parallel. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors (see angle θ in the figure).



Example 7. (a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$

(b) Find symmetric equations for the line of intersection L of these two planes.

Example 8. Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

Example 9. Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

Example 10. In Example 3 we showed that the lines

$$\begin{array}{llll} L_1: & x = 1 + t & y = -2 + 3t & z = 4 - t \\ L_2: & x = 2s & y = 3 + s & z = -3 + 4s \end{array}$$

are skew. Find the distance between them.

12.6 Cylinders and Quadric Surfaces

Definition 12.6.1. The curves of intersection of a surface with planes parallel to the coordinate planes are called traces (or cross-sections) of the surface.

Definition 12.6.2. A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.

Example 1. Sketch the graph of the surface $z = x^2$.

Example 2. Identify and sketch the surfaces.

(a) $x^2 + y^2 = 1$

(b) $y^2 + z^2 = 1$

Definition 12.6.3. A quadric surface is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, B, C, \ldots, J$ are constants, but by translation and rotation it can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0.$$

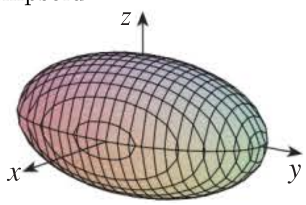
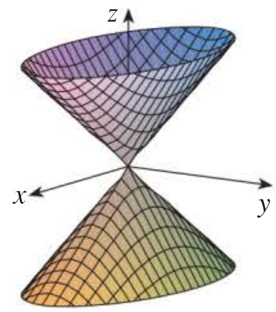
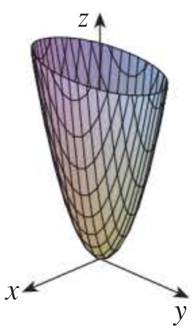
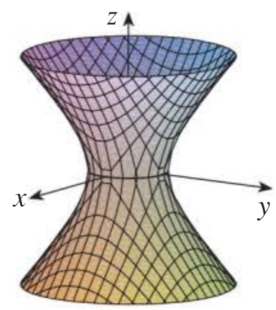
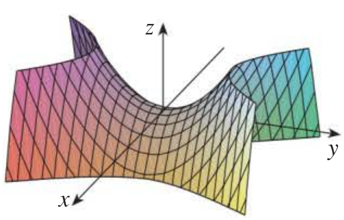
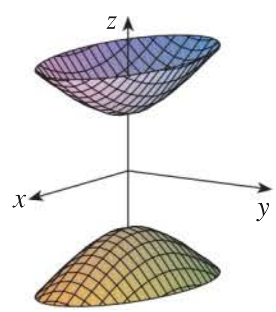
Example 3. Use traces to sketch the quadric surface with equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1.$$

Example 4. Use traces to sketch the surface $z = 4x^2 + y^2$.

Example 5. Sketch the surface $z = y^2 - x^2$.

Example 6. Sketch the surface $\dfrac{x^2}{4} + y^2 - \dfrac{z^2}{4} = 1$.

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Example 7. Identify and sketch the surface $4x^2 - y^2 + 2z^2 + 4 = 0$.

Example 8. Classify the quadric surface $x^2 + 2z^2 - 6x - y + 10 = 0$.

Chapter 13

Vector Functions

13.1 Vector Functions and Space Curves

Definition 13.1.1. A vector-valued function, or vector function is a function whose domain is a set of real numbers and whose range is a set of vectors. If $f(t)$, $g(t)$, and $h(t)$ are the components of a vector function $\mathbf{r}(t)$ whose values are three-dimensional vectors, then we call f, g, and h the component functions of $\mathbf{r}$ and we can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

Example 1. What are the component functions and domain of

$$\mathbf{r}(t) = \left\langle t^3, \ln(3-t), \sqrt{t} \right\rangle?$$

Definition 13.1.2. The limit of a vector function $\mathbf{r}$ is defined by taking the limits of its component functions, i.e., if $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

Example 2. Find $\lim_{t \to 0} \mathbf{r}(t)$, where $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\mathbf{j} + \frac{\sin t}{t}\mathbf{k}$.

Definition 13.1.3. A vector function $\mathbf{r}$ is <u>continuous at a</u> if

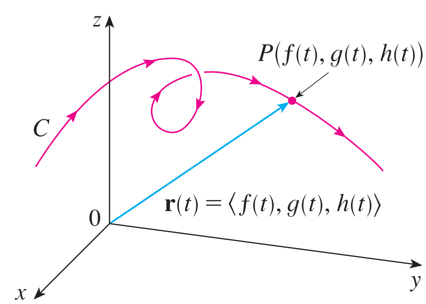
$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a),$$

so $\mathbf{r}$ is continuous at a if and only if its component functions f, g, and h are continuous at a.

Definition 13.1.4. Suppose that f, g, and h are continuous real-valued functions on an interval I. Then the set C of all points (x, y, z) in space, where

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

(called the <u>parametric equations of C</u> for a parameter t) and t varies throughout the interval I, is called a <u>space curve</u>. If we consider the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\mathbf{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on C. Thus any continuous vector function $\mathbf{r}$ defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$, as shown in the figure.



Example 3. Describe the curve defined by the vector function

$$\mathbf{r}(t) = \langle 1 + t, 2 + 5t, -1 + 6t \rangle.$$

Example 4. Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}.$$

Example 5. Find a vector equation and parametric equations for the line segment that joins the point $P(1, 3, -2)$ to the point $Q(2, -1, 3)$.

Example 6. Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

Example 7. Use a computer to draw the curve with vector equation $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$. This curve is called a twisted cubic.

13.2 Vector Function Derivatives & Integrals

Definition 13.2.1. The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined as

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists.

Definition 13.2.2. The vector $\mathbf{r}'(t)$ is called the tangent vector to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$. The tangent line to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Theorem 13.2.1. *If* $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, *where* f, g, *and* h *are differentiable functions, then*

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Proof.

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}[\mathbf{r}(t+\Delta t) - \mathbf{r}(t)] \\
&= \lim_{\Delta t\to 0} \frac{1}{\Delta t}[\langle f(t+\Delta t), g(t+\Delta t), h(t+\Delta t)\rangle - \langle f(t), g(t), h(t)\rangle] \\
&= \lim_{\Delta t\to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle \\
&= \left\langle \lim_{\Delta t\to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}, \lim_{\Delta t\to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}, \lim_{\Delta t\to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle \\
&= \langle f'(t), g'(t), h'(t)\rangle.
\end{aligned}$$

□

Example 1. (a) Find the derivative of $\mathbf{r}(t) = (1+t^3)\mathbf{i} + te^{-t}\mathbf{j} + \sin 2t\mathbf{k}$.

(b) Find the unit tangent vector at the point where $t = 0$.

Example 2. For the curve $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (2 - t)\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$.

Example 3. Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $(0, 1, \pi/2)$.

Definition 13.2.3. The <u>second derivative</u> of a vector function $\mathbf{r}$ is the derivative of $\mathbf{r}'$, that is, $\mathbf{r}'' = (\mathbf{r}')'$.

Theorem 13.2.2. *Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are differentiable vector functions,* c *is a scalar, and* f *is a real-valued function. Then*

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Example 4. Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

Definition 13.2.4. The <u>definite integral</u> of a continuous vector function $\mathbf{r}(t)$ is

$$\begin{aligned}\int_a^b \mathbf{r}(t)dt &= \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{r}(t_i^*)\Delta t \\ &= \lim_{n\to\infty} \left[\left(\sum_{i=1}^{n} f(t_i^*)\Delta t \right) \mathbf{i} + \left(\sum_{i=1}^{n} g(t_i^*)\Delta t \right) \mathbf{j} + \left(\sum_{i=1}^{n} h(t_i^*)\Delta t \right) \mathbf{k} \right]\end{aligned}$$

and so

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt \right) \mathbf{i} + \left(\int_a^b g(t)dt \right) \mathbf{j} + \left(\int_a^b h(t)dt \right) \mathbf{k}.$$

Theorem 13.2.3. *We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:*

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a).$$

where $\mathbf{R}$ *is an antiderivative of* $\mathbf{r}$*, that is,* $\mathbf{R}'(t) = \mathbf{r}(t)$*. We use the notation* $\int \mathbf{r}(t)dt$ *for indefinite integrals (antiderivatives).*

Example 5. If $\mathbf{r}(t) = 2\cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$, then what are $\int \mathbf{r}(t)dt$ and $\int_0^{\pi/2} \mathbf{r}(t)dt$?

13.3 Arc Length and Curvature

Definition 13.3.1. If a space curve is given by $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, $a \leq t \leq b$, or equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f', g', and h' are continuous, then the length of the curve traversed exactly once as t increases from a to b is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt$$
$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt,$$

or equivalently,

$$L = \int_a^b |\mathbf{r}'(t)|dt.$$

Example 1. Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

Remark 1. A single curve C can be represented by more than one vector function. For instance, the twisted cubic

$$\mathbf{r}_1(t) = \langle t, t^2, t^3\rangle \qquad 1 \leq t \leq 2$$

could also be represented by the function

$$\mathbf{r}_2(t) = \langle e^u, e^{2u}, e^{3u}\rangle \qquad 0 \leq u \leq \ln 2$$

We say that these equations are parametrizations of the curve C. It can be shown that our arc length equation is independent of the parametrization that is used.

Definition 13.3.2. Suppose that C is a curve given by a vector function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \qquad a \leq t \leq b$$

where $\mathbf{r}'$ is continuous and C is traversed exactly once as t increases from a to b. We define its arc length function s by

$$s(t) = \int_a^t |\mathbf{r}'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du$$

where differentiating both sides of the arc length function using the Fundamental Theorem of Calculus gives

$$\frac{ds}{dt} = |\mathbf{r}'(t)|.$$

Remark 2. If a curve $\mathbf{r}(t)$ is already given in terms of a parameter t and $s(t)$ is the arc length function, then we may be able to solve for t as a function of s: $t = t(s)$. Then the curve can be reparametrized with respect to arc length by substituting for t: $\mathbf{r} = \mathbf{r}(t(s))$.

Example 2. Reparametrize the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

Definition 13.3.3. The curvature of a curve C at a given point is a measure of how quickly the curve changes direction at that point, defined as

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$

where $\mathbf{T}$ is the unit tangent vector.

Remark 3. A parametrization is called smooth on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on I. A curve is called smooth if it has a smooth parametrization. Since the unit tangent vector is only defined for smooth curves, the curvature is only defined for smooth curves.

Theorem 13.3.1.

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

Proof. By the chain rule

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt},$$

so

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

□

Example 3. Show that the curvature of a circle of radius a is $1/a$.

Theorem 13.3.2. *The curvature of the curve given by the vector function* $\mathbf{r}$ *is*

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Proof. Since $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ and $|\mathbf{r}'| = ds/dt$, we have

$$\begin{aligned} \mathbf{r}' &= |\mathbf{r}'|\mathbf{T} = \frac{ds}{dt}\mathbf{T} \\ \mathbf{r}'' &= \frac{d^2 s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'. \end{aligned}$$

Since $\mathbf{T} \times \mathbf{T} = \mathbf{0}$, we have

$$\begin{aligned} \mathbf{r}' \times \mathbf{r}'' &= \frac{ds}{dt}\mathbf{T} \times \left(\frac{d^2 s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}' \right) \\ \mathbf{r}' \times \mathbf{r}'' &= \frac{ds}{dt}\mathbf{T} \times \frac{d^2 s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T} \times \frac{ds}{dt}\mathbf{T}' \\ \mathbf{r}' \times \mathbf{r}'' &= \left(\frac{ds}{dt}\frac{d^2 s}{dt^2} \right)(\mathbf{T} \times \mathbf{T}) + \left(\frac{ds}{dt} \right)^2 (\mathbf{T} \times \mathbf{T}') \\ \mathbf{r}' \times \mathbf{r}'' &= \left(\frac{ds}{dt} \right)^2 (\mathbf{T} \times \mathbf{T}'). \end{aligned}$$

Since $|\mathbf{T}(t)| = 1$ for all t, $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal, so

$$\begin{aligned} |\mathbf{r}' \times \mathbf{r}''| &= \left(\frac{ds}{dt} \right)^2 |\mathbf{T} \times \mathbf{T}'| \\ &= \left(\frac{ds}{dt} \right)^2 |\mathbf{T}||\mathbf{T}'| \sin\left(\frac{\pi}{2} \right) \\ &= \left(\frac{ds}{dt} \right)^2 |\mathbf{T}'|. \end{aligned}$$

Thus

$$|\mathbf{T}'| = \frac{|\mathbf{r}' \times \mathbf{r}''|}{(ds/dt)^2} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^2}$$

and

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}.$$

□

Example 4. Find the curvature of the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at a general point and at $(0, 0, 0)$.

Theorem 13.3.3. *If $y = f(x)$ is a plane curve, then*

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

Proof. Choose x as the parameter and write $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j}$. Then $\mathbf{r}'(x) = \mathbf{i} + f'(x)\mathbf{j}$ and $\mathbf{r}''(x) = f''(x)\mathbf{j}$. Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, it follows that $\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x)\mathbf{k}$. We also have $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$, and so

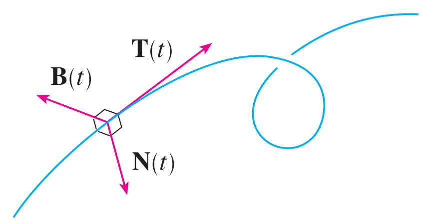
$$\kappa(x) = \frac{|\mathbf{r}'(x) \times \mathbf{r}''(x)|}{|\mathbf{r}'(x)|^3} = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

□

Example 5. Find the curvature of the parabola $y = x^2$ at the points $(0, 0)$, $(1, 1)$, and $(2, 4)$.

Definition 13.3.4. For any point where $\kappa \neq 0$, the principal unit normal vector $\mathbf{N}(t)$ (or simply unit normal) is defined to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|},$$



and so it is orthogonal to the unit tangent vector $\mathbf{T}(t)$. The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the binormal vector. It is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and is also a unit vector. (See the figure.)

Example 6. Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}.$$

Definition 13.3.5. The plane determined by the normal and binormal vectors **N** and **B** at point P on a curve C is called the normal plane of C at P. It consists of all lines that are orthogonal to the tangent vector **T**. The plane determined by the vectors **T** and **N** is called the osculating plane of C at P. It is the plane that comes closest to containing the part of the curve near P.

Definition 13.3.6. The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which **N** points), and has radius $\rho = 1/\kappa$ (the reciprocal of the curvature) is called the osculating circle (or the circle of curvature) of C at P. It is the circle that best describes how C behaves near P; it shares the same tangent, normal, and curvature at P.

Example 7. Find equations of the normal plane and osculating plane of the helix in Example 6 at the point $P(0, 1, \pi/2)$.

Example 8. Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

13.4 Motion in Space

Definition 13.4.1. Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$. Then the velocity vector $\mathbf{v}(t)$ at time t is given by

$$\mathbf{v}(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t).$$

The speed of the particle at time t is the magnitude of the velocity vector, that is, $|\mathbf{v}(t)|$. As in the case of one-dimensional motion, the acceleration of the particle is defined as the derivative of the velocity:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).$$

Example 1. The position vector of an object moving in a plane is given by $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$. Find its velocity, speed, and acceleration when $t = 1$ and illustrate geometrically.

Example 2. Find the velocity, acceleration, and speed of a particle with position vector $\mathbf{r}(t) = \langle t^2, e^t, te^t \rangle$.

Example 3. A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 4t\mathbf{i} + 6t\mathbf{j} + \mathbf{k}$. Find its velocity and position at time t.

Remark 1. In general, vector integrals allow us to recover velocity when acceleration is known and position when velocity is known:

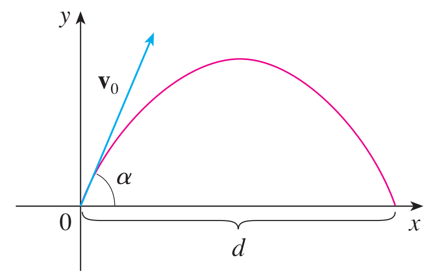
$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(u)du \qquad \mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(u)du.$$

If the force that acts on a particle is known, then the acceleration can be found from Newton's Second Law of Motion. The vector version of this law states that if, at any time t, a force $\mathbf{F}(t)$ acts on an object of mass m producing an acceleration $\mathbf{a}(t)$, then

$$\mathbf{F}(t) = m\mathbf{a}(t).$$

Example 4. An object with mass m that moves in a circular path with constant angular speed ω has position vector $\mathbf{r}(t) = a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j}$. Find the force acting on the object and show that it is directed toward the origin.

Example 5. A projectile is fired with angle of elevation α and initial velocity $\mathbf{v}_0$. (See the figure.) Assuming that air resistance is negligible and the only external force is due to gravity, find the position function $\mathbf{r}(t)$ of the projectile. What value of α maximizes the range (the horizontal distance traveled)?



Example 6. A projectile is fired with muzzle speed 150 m/s and angle of elevation 45° from a position 10 m above ground level. Where does the projectile hit the ground, and with what speed?

Theorem 13.4.1. *If $v = |\mathbf{v}|$ is the speed of a particle in motion, then*

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

where $a_T = v'$ and $a_N = \kappa v^2$.

Proof.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{v}}{v},$$

so

$$\begin{aligned} \mathbf{v} &= v\mathbf{T} \\ \mathbf{a} = \mathbf{v}' &= v'\mathbf{T} + v\mathbf{T}'. \end{aligned}$$

By our expression for curvature,

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v},$$

so $|\mathbf{T}'| = \kappa v$. Since $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$,

$$\mathbf{T}' = |\mathbf{T}'|\mathbf{N} = \kappa v\mathbf{N},$$

and thus

$$\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$$

□

Theorem 13.4.2.

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} \qquad a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}.$$

Proof.

$$\begin{aligned} \mathbf{v} \cdot \mathbf{a} &= v\mathbf{T} \cdot (v'\mathbf{T} + \kappa v^2\mathbf{N}) \\ &= vv'\mathbf{T} \cdot \mathbf{T} + \kappa v^3\mathbf{T} \cdot \mathbf{N} \\ &= vv', \end{aligned}$$

so

$$\begin{aligned} a_T &= v' = \frac{\mathbf{v} \cdot \mathbf{a}}{v} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} \\ a_N &= \kappa v^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}|\mathbf{r}'(t)|^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}. \end{aligned}$$

□

Example 7. A particle moves with position function $\mathbf{r}(t) = \langle t^2, t^2, t^3 \rangle$. Find the tangential and normal components of acceleration.

Chapter 14

Partial Derivatives

14.1 Functions of Several Variables

Definition 14.1.1. A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

Remark 1. We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are independent variables and z is the dependent variable.

Example 1. For each of the following functions, evaluate $f(3, 2)$ and find and sketch the domain.

(a) $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$

(b) $f(x, y) = x\ln(y^2 - x)$

Example 2. In regions with severe winter weather, the *wind-chill index* is often used to describe the apparent severity of the cold. This index W is a subjective temperature that depends on the actual temperature T and the wind speed v. So W is a function of T and v, and we can write $W = f(T, v)$. The table records values of W compiled by the US National Weather Service and the Meteorological Service of Canada.

Wind-chill index as a function of air temperature and wind speed

Wind speed (km/h)

Actual temperature (°C)

T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

Find $f(-5, 50)$ and interpret its meaning in context.

Example 3. In 1928 Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899-1922. They considered a simplified view of the economy in which production output is determined by the amount of labor involved and the amount of capital invested. While there are many other factors affecting economic performance, their model proved to be remarkably accurate. The function they used to model production was of the form

$$P(L, K) = bL^{\alpha}K^{1-\alpha},$$

known as the Cobb-Douglas production function, where P is the total production (the monetary value of all goods produced in a year), L is the amount of labor (the total number of person-hours worked in a year), and K is the amount of capital invested (the monetary worth of all machinery, equipment, and buildings).
Cobb and Douglas used economic data published by the government to obtain the table on the right. They took the year 1899 as a baseline and P, L, and K for 1899 were each assigned the value 100. The values for other years were expressed as percentages of the 1899 figures.
Cobb and Douglas used the method of least squares to fit the data of the table to the function

$$P(L, K) = 1.01L^{0.75}K^{0.25}.$$

Use this function to compute the production in the years 1910 and 1920, and compare your results with the actual values for these years.

Year	P	L	K
1899	100	100	100
1900	101	105	107
1901	112	110	114
1902	122	117	122
1903	124	122	131
1904	122	121	138
1905	143	125	149
1906	152	134	163
1907	151	140	176
1908	126	123	185
1909	155	143	198
1910	159	147	208
1911	153	148	216
1912	177	155	226
1913	184	156	236
1914	169	152	244
1915	189	156	246
1916	225	183	298
1917	227	198	335
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

Example 4. Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Definition 14.1.2. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Definition 14.1.3. The level curves of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

Example 5. Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

Definition 14.1.4. The function

$$f(x, y) = ax + by + c$$

is called a linear function. The graph of such a function has the equation

$$z = ax + by + c \qquad \text{or} \qquad ax + by - z + c = 0,$$

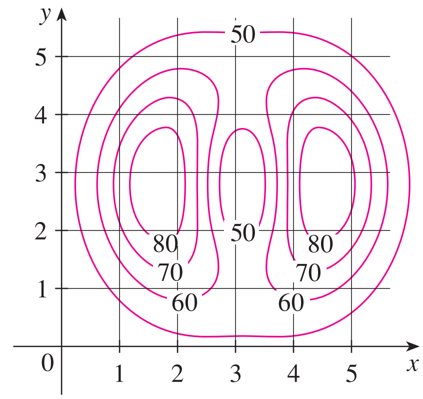
so it is a plane.

Example 6. Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Example 7. Use a computer to draw the graph of the Cobb-Douglas production function $P(L, K) = 1.01L^{0.75}K^{0.25}$.

Example 8. Find the domain and range and sketch the graph of $h(x, y) = 4x^2 + y^2$.

Example 9. A contour map for a function f is shown in the figure. Use it to estimate the values of $f(1,3)$ and $f(4,5)$.



Example 10. Sketch the level curves of the function $f(x,y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

Example 11. Sketch the level curves of the function

$$g(x,y) = \sqrt{9 - x^2 - y^2} \qquad \text{for} \quad k = 0, 1, 2, 3.$$

Example 12. Sketch some level curves of the function $h(x, y) = 4x^2 + y^2 + 1$.

Example 13. Plot level curves for the Cobb-Douglas production function of Example 3.

Definition 14.1.5. A function of three variables, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.

Example 14. Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z.$$

Definition 14.1.6. The level surfaces of a function f of three variables are the curves with equations $f(x, y, z) = k$, where k is a constant.

Example 15. Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2.$$

Definition 14.1.7. A function of n variables is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples.

Remark 2. Sometimes we will use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we will often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \ldots, x_n)$.

14.2 Limits and Continuity

Definition 14.2.1. Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \varepsilon$.

Remark 1. If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Example 1. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Example 2. If $f(x, y) = xy/(x^2 + y^2)$, does $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ exist?

Example 3. If $f(x,y) = \dfrac{xy^2}{x^2+y^4}$, does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

Remark 2. The Limit Laws listed in section 2.3 can be extended to functions of two variables: the limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and so on. In particular, the following equations are true.

$$\lim_{(x,y)\to(a,b)} x = a \qquad \lim_{(x,y)\to(a,b)} y = b \qquad \lim_{(x,y)\to(a,b)} c = c.$$

The Squeeze Theorem also holds.

Example 4. Find $\lim_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2}$ if it exists.

Definition 14.2.2. A function f of two variables is called <u>continuous at (a,b)</u> if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

We say that f is <u>continuous on D</u> if f is continuous at every point (a,b) in D.

Definition 14.2.3. A <u>polynomial of two variables</u> (or polynomial, for short) is a sum of terms of the form cx^my^n, where c is a constant and m and n are nonnegative integers. A <u>rational function</u> is a ratio of polynomials.

Remark 3. The limits in Remark 2 show that the functions $f(x,y) = x$, $g(x,y) = y$, and $h(x,y) = c$ are continuous. Since any polynomial can be built up out of the simple functions f, g, and h by multiplication and addition, it follows that all polynomials are continuous on $\mathbb{R}^2$. Likewise, any rational function is continuous on its domain because it is a quotient of continuous functions.

Example 5. Evaluate $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

Example 6. Where is the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

Example 7. Where is the function

$$g(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

continuous?

Remark 4. If f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

Example 8. Where is the function

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous?

Example 9. Where is the function $h(x,y) = \arctan(y/x)$ continuous?

Definition 14.2.4. The notation

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$$

means that the values of $f(x,y,z)$ approach the number L as the point (x,y,z) approaches the point (a,b,c) along any path in the domain of f. Precisely, for every number $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that if $f(x,y,z)$ is in the domain of f and $0 < \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} < \delta$ then $|f(x,y,z) - L| < \varepsilon$. The function is continuous at (a,b,c) if

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c).$$

Definition 14.2.5. If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $\mathbf{x} \in D$ and $0 < |\mathbf{x} - \mathbf{a}| < \delta$ then $|f(\mathbf{x}) - L| < \varepsilon$. The function is continuous at $\mathbf{a}$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).$$

14.3 Partial Derivatives

Definition 14.3.1. In general, if f is a function of two variables x and y, suppose we only let x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are considering a function of a single variable x, say $g(x) = f(x, b)$. If g has a derivative at a, then we call it the partial derivative of f with respect to x at (a, b) and denote it by $f_x(a, b)$. Thus

$$f_x(a,b) = g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

Similarly, the partial derivative of f with respect to y at (a, b), denoted by $f_y(a, b)$, is obtained by keeping x fixed ($x = a$) and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

Definition 14.3.2. If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ f_y(x,y) &= \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}. \end{aligned}$$

Definition 14.3.3 (Notations for Partial Derivatives)**.** If $z = f(x, y)$, we write

$$\begin{aligned} f_x(x,y) &= f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f \\ f_y(x,y) &= f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f. \end{aligned}$$

Remark 1 (Rule for Finding Partial Derivatives of $z = f(x, y)$).

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example 1. If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Example 2. If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

Example 3. The body mass index of a person is defined by

$$B(m, h) = \frac{m}{h^2}.$$

Calculate the partial derivatives of B for a young man with $m = 64$ kg and $h = 1.68$ m and interpret them.

Example 4. If $f(x,y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Example 5. Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1.$$

Definition 14.3.4. If f is a function of three variables x, y and z, then its partial derivative with respect to x is defined as

$$f_x(x,y,z) = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

and it is found by regarding y and z as constants and differentiating $f(x,y,z)$ with respect to x.

Definition 14.3.5. In general, if u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the ith variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1,\ldots,x_{i-1},x_i+h,x_{i+1},\ldots,x_n) - f(x_1,\ldots,x_i,\ldots,x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f.$$

Example 6. Find f_x, f_y, and f_z if $f(x,y,z) = e^{xy}\ln z$.

Definition 14.3.6. If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the second partial derivatives of f. If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$
$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$
$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$
$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}.$$

Thus the notation f_{xy} (or $\partial^2 f/\partial y \partial x$) means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

Example 7. Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2.$$

Theorem 14.3.1 (Clairaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Remark 2. Partial derivatives of order 3 or higher can also be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y^2 \partial x}$$

and using Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

Example 8. Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

Definition 14.3.7. The partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called Laplace's equation. Solutions of this equation are called harmonic functions.

Example 9. Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

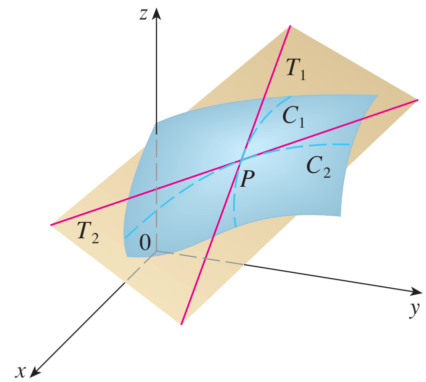
Definition 14.3.8. The <u>wave equation</u>

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibration string.

Example 10. Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

14.4 Tangent Planes & Linear Approximations



Definition 14.4.1. Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S, so that P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. Then the tangent plane to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2. (See the figure.)

Theorem 14.4.1. *Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is*

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Proof. Any line passing through P has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

By dividing this equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

$$z - z_0 = a(x - x_0) + b(y - y_0).$$

If this equation represents the tangent plane at P, then its intersection with the tangent line $y = y_0$ must be T_1, so by letting $y = y_0$ we get

$$z - z_0 = a(x - x_0)$$

as the equation of T_1, and since T_1 has slope $f_x(x_0, y_0)$, we have $a = f_x(x_0, y_0)$. Similarly, by letting $x = x_0$, we get $z - z_0 = b(y - y_0)$ as the equation of T_2, so $b = f_y(x_0, y_0)$. □

Example 1. Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Definition 14.4.2. The linear function whose graph is the tangent plane at the point to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is called the linearization of f at (a, b) and is given by

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linear approximation or the tangent line approximation of f at (a, b).

Definition 14.4.3. Suppose $z = f(x, y)$ is a function of two variables where x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. Then the corresponding increment of z is

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b).$$

Definition 14.4.4. If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Theorem 14.4.2. *If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).*

Example 2. Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Example 3. On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates. The National Weather Service has devised the *heat index* (also called the temperature-humidity index, or humidex, in some countries) to describe the combined effects of temperature and humidity. The heat index I is the perceived air temperature when the actual temperature is T and the relative humidity is H. So I is a function of T and H and we can write $I = f(T, H)$. The following table of values of I is an excerpt from a table compiled by the National Weather Service.

Heat index I as a function of temperature and humidity

Relative humidity (%)

Actual temperature (°F)

T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Find a linear approximation for the heat index $I = f(T, H)$ when T is near 96°F and H is near 70%. Use it to estimate the heat index when the temperature is 97°F and the relative humidity is 72%.

Definition 14.4.5. For a differentiable function of two variables, $z = f(x, y)$, we define the differentials dx and dy to be independent variables; that is, they can be given any values. Then the differential dz, also called the total differential, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Example 4.

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

Example 5. The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

Remark 1. Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables. A differentiable function is defined by an expression similar to the one in Definition 14.4.4. For such functions the linear approximation is

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

and the linearization $L(x,y,z)$ is the right side of this expression.
If $w = f(x,y,z)$ then the increment of w is

$$\Delta w = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z).$$

The differential dw is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz.$$

Example 6. The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

14.5 The Chain Rule

Theorem 14.5.1 (The Chain Rule (Case 1))**.** *Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are differentiable functions of t. Then z is a differentiable function of t and*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Proof.

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Dividing both sides of this equation by Δt, we have

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}.$$

If we now let $\Delta t \to 0$, then $\Delta x = g(t+\Delta t) - g(t) \to 0$ because g is differentiable and therefore continuous. Similarly, $\Delta y \to 0$. This, in turn, means that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, so

$$\begin{aligned}
\frac{dz}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} \\
&= \frac{\partial f}{\partial x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} + \left(\lim_{\Delta t \to 0} \varepsilon_1\right) \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \left(\lim_{\Delta t \to 0} \varepsilon_2\right) \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt} \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.
\end{aligned}$$

□

Example 1. If $z = x^2 y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

Example 2. The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

Theorem 14.5.2 (The Chain Rule (Case 2))**.** *Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Example 3. If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

Theorem 14.5.3 (The Chain Rule (General Version)). *Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and*

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

Example 4. Write out the Chain Rule for the case where $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$.

Example 5. If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, $t = 0$.

Example 6. If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

Example 7. If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find

(a) $\partial z/\partial r$

(b) $\partial^2 z/\partial r^2$

Theorem 14.5.4 (Implicit Differentiation)**.** *Suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f. If F is differentiable,*

$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}.$$

Proof. If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x to get

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

But $dy/dx = 1$, so if $\partial F/\partial y \neq 0$ we can solve for dy/dx and obtain the desired result. □

Example 8. Find y' if $x^3 + y^3 = 6xy$.

Theorem 14.5.5. *Suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable,*

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

Example 9. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

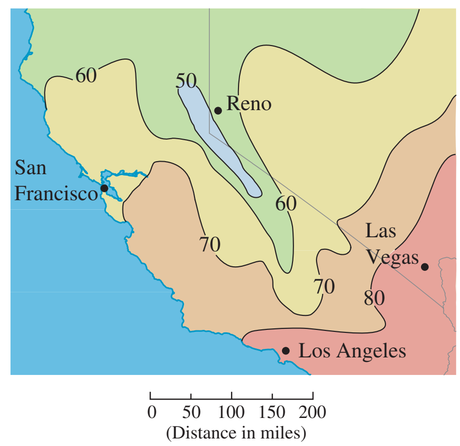
14.6 Directional Derivatives and the Gradient

Definition 14.6.1. The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h},$$

if this limit exists.

Example 1. Use the weather map in the figure to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.



Theorem 14.6.1. *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and*

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

Proof. If we define a function g of the single variable h by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of the derivative, we have

$$\begin{aligned} g'(0) &= \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \\ &= D_{\mathbf{u}} f(x_0, y_0). \end{aligned}$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0+ha$, $y = y_0+hb$, so the Chain Rule gives

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)a + f_y(x, y)b.$$

If we now put $h = 0$, then $x = x_0$, $y = y_0$, and

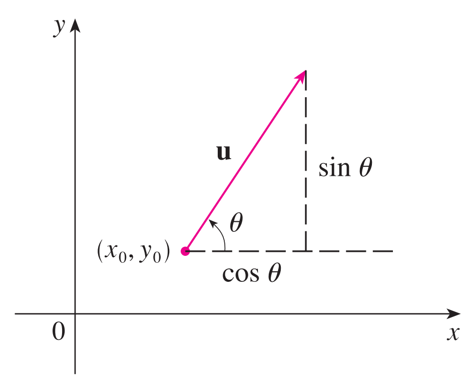
$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b.$$

Thus

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b. \qquad \square$$

Remark 1. If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis (as in the figure), then we can write $\mathbf{u} = \langle\cos\theta, \sin\theta\rangle$ and the formula in Theorem 14.6.1 becomes

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$



Example 2. Find the directional derivative $D_{\mathbf{u}}f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}}f(1, 2)$?

Definition 14.6.2. If f is a function of two variables x and y, then the gradient of f is the vector function ∇f (or $\mathbf{grad}f$) defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Example 3. If $f(x, y) = \sin x + e^{xy}$, then find $\nabla f(x, y)$ and $\nabla f(0, 1)$.

Remark 2. With this notation for the gradient vector, we can rewrite the equation for the directional derivative of a differentiable function as

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

This expresses the directional derivative in the direction of a unit vector $\mathbf{u}$ as the scalar projection of the gradient vector onto $\mathbf{u}$.

Example 4. Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

Definition 14.6.3. The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists. More compactly,

$$D_{\mathbf{u}}f(\mathbf{x}_0) = \lim_{h\to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where $\mathbf{x}_0 = \langle x_0, y_0\rangle$ if $n = 2$ and $\mathbf{x}_0 = \langle x_0, y_0, z_0\rangle$ if $n = 3$.

Remark 3. If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c\rangle$, then the same method that was used to prove Theorem 14.6.1 can be used to show that

$$D_{\mathbf{u}}f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c.$$

For a function of three variables, the gradient vector, denoted by ∇f or $\mathbf{grad} f$, is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle,$$

or, for short,

$$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Just as with functions of two variables, the directional derivative can be rewritten as

$$D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}.$$

Example 5. If $f(x, y, z) = x \sin yz$,

(a) find the gradient of f

(b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Theorem 14.6.2. *Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.*

Proof.

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta$$

where θ is the angle between ∇f and $\mathbf{u}$. The maximum value of $\cos\theta$ is 1 and this occurs when $\theta = 0$. Therefore the maximum value of $D_{\mathbf{u}}f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when $\mathbf{u}$ has the same direction as ∇f. □

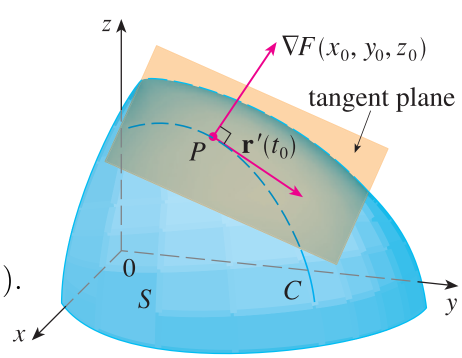
Example 6.

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q\left(\frac{1}{2}, 2\right)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Example 7. Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z, in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

Definition 14.6.4. If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, the tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ is the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. (See the figure.) Using the standard equation of a plane, we can write the equation of this tangent plane as



$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(x - x_0).$$

Definition 14.6.5. The normal line to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so its symmetric equations are

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}.$$

Example 8. Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

14.7 Maximum and Minimum Values

Definition 14.7.1. A function of two variables has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). The number $f(a, b)$ is called a local maximum value. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a local minimum at (a, b) and $f(a, b)$ is a local minimum value. If these inequalities hold for all points (x, y) in the domain of f, then f has an absolute maximum (or absolute minimum) at (a, b).

Theorem 14.7.1. *If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.*

Proof. Let $g(x) = f(x, b)$. If f has a local maximum (or minimum) at (a, b), then g has a local maximum (or minimum) at a, so $g'(a) = 0$ by Fermat's Theorem. But $g'(a) = f_x(a, b)$ and so $f_x(a, b) = 0$. Similarly, by applying Fermat's Theorem to the function $G(y) = f(a, y)$, we obtain $f_y(a, b) = 0$. □

Definition 14.7.2. A point (a, b) is called a critical point (or stationary point) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

Example 1. Find the extreme values of $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

Example 2. Find the extreme values of $f(x, y) = y^2 - x^2$.

Theorem 14.7.2 (Second Derivatives Test)**.** *Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let*

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

Remark 1. In case (c) the point (a, b) is called a saddle point of f and the graph of f crosses its tangent plane at (a, b).

Remark 2. If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

Remark 3. To remember the formula for D, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2.$$

Example 3. Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

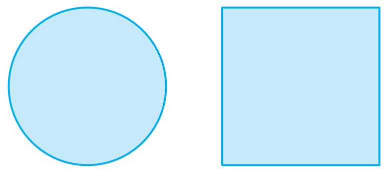
Example 4. Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4.$$

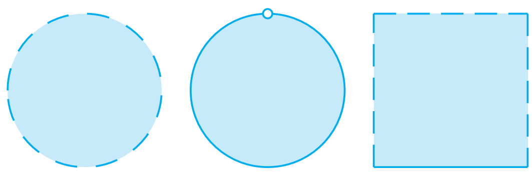
Example 5. Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

Example 6. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Definition 14.7.3. A closed set in $\mathbb{R}^2$ is one that contains all its boundary points. [A boundary point of D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D.]
A bounded set in $\mathbb{R}^2$ is one that is contained within some disk.



Closed sets



Sets that are not closed

Theorem 14.7.3 (Extreme Value Theorem for Functions of Two Variables)**.** *If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.*

Remark 4. To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 7. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

14.8 Lagrange Multipliers

Theorem 14.8.1 (Method of Lagrange Multipliers)**.** *To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:*

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k.$$

The number λ is called a Lagrange multiplier.

(b) Evaluate f at the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Example 1. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Example 2. Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Example 3. Find the extreme values of $f(x, y) = x^2+2y^2$ on the disk $x^2+y^2 \leq 1$.

Example 4. Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

Theorem 14.8.2 (Method of Lagrange Multipliers for Two Constraints)**.** *To find the maximum and minimum values of $f(x, y, z)$ subject to the constraints $g(x, y, z) = k$ and $h(x, y, z) = c$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$, $\nabla h \neq \mathbf{0}$, and ∇g is not parallel to ∇h]:*

(a) Find all values of x, y, z, λ, and μ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$

and

$$g(x, y, z) = k \qquad h(x, y, z) = c.$$

The numbers λ and μ are called Lagrange multipliers.

(b) Evaluate f at the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Example 5. Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Chapter 15

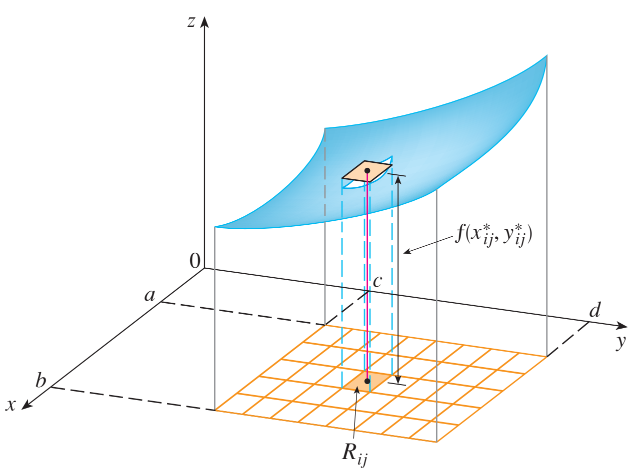
Multiple Integrals

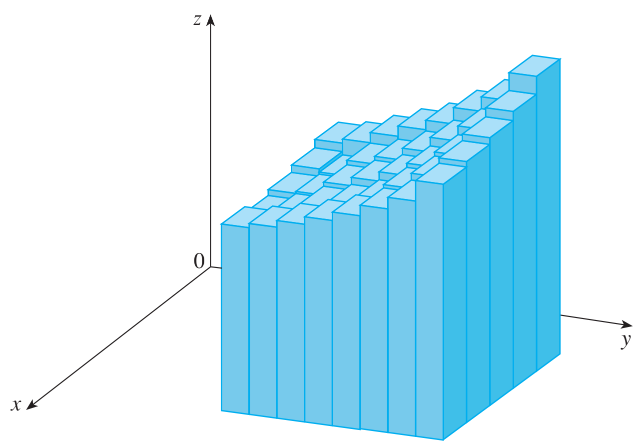
15.1 Double Integrals over Rectangles

Definition 15.1.1. The double integral of f over the rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

if this limit exists. The points (x_{ij}^*, y_{ij}^*) are called sample points, $\Delta A = \Delta x\, \Delta y$ is the area of the subrectangle R_{ij} formed by the subintervals $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$, and the sum is called a double Riemann sum.





Definition 15.1.2. If $f(x, y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA.$$

Example 1. Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

Example 2. If $R = \{(x, y) \mid -1 \leq x \leq 1, -2 \leq y \leq 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA.$$

Theorem 15.1.1 (Midpoint Rule for Double Integrals)**.**

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_j)\,\Delta A$$

where $\bar{x}_i$ *is the midpoint of* $[x_{i-1}, x_i]$ *and* $\bar{y}_i$ *is the midpoint of* $[y_{j-1}, y_j]$.

Example 3. Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x - 3y^2)\,dA$, where $R = \{(x,y) \mid 0 \le x \le 2, 1 \le y \le 2\}$.

Definition 15.1.3. Suppose that f is a function of two variables that is integrable on the rectangle $R = [a,b] \times [c,d]$. We use the notation $\int_a^b f(x,y)dx$ to mean that y is held fixed and $f(x,y)$ is integrated with respect to x from $x = a$ to $x = b$. This procedure is called partial integration with respect to x. Integrating this function gives us an <u>iterated integral</u>

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

where we first integrate with respect to x (holding y fixed) from $x = a$ to $x = b$ and then we integrate the resulting function of y with respect to y from $y = c$ to $y = d$.

Example 4. Evaluate the iterated integrals.

(a) $\displaystyle\int_0^3 \int_1^2 x^2 y \, dy \, dx$

(b) $\displaystyle\int_1^2 \int_0^3 x^2 y \, dx \, dy$

Theorem 15.1.2 (Fubini's Theorem)**.** *If f is continuous on the rectangle $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$, then*

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy.$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 5. Evaluate the double integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$.

Example 6. Evaluate $\iint_R y \sin(xy)\, dA$, where $R = [1, 2] \times [0, \pi]$.

Example 7. Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

Theorem 15.1.3.

$$\iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy \qquad \textit{where } R = [a,b]\times[c,d].$$

Proof. By Fubini's Theorem,

$$\iint_R g(x)h(y)\,dA = \int_c^d \int_a^b g(x)h(y)\,dx\,dy = \int_c^d \left[\int_a^b g(x)h(y)\,dx\right] dy.$$

In the inner integral, y is a constant, so $h(y)$ is a constant and we can write

$$\int_c^d \left[\int_a^b g(x)h(y)\,dx\right] dy = \int_c^d \left[h(y)\left(\int_a^b g(x)\,dx\right)\right] dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

since $\int_a^b g(x)\,dx$ is a constant. □

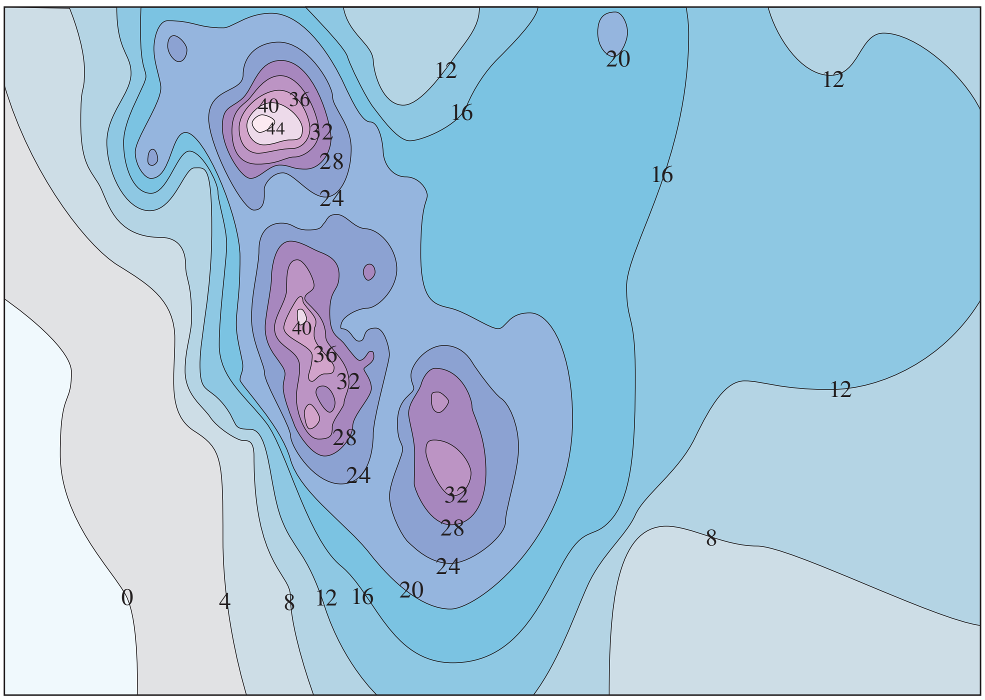
Example 8. Find $\iint_R \sin x \cos y\,dA$ if $R = [0,\pi/2]\times[0,\pi/2]$.

Definition 15.1.4. The <u>average value</u> of a function f of two variables defined on a rectangle R is

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)\,dA$$

where $A(R)$ is the area of R.

Example 9. The contour map in the figure shows the snowfall, in inches, that fell on the state of Colorado on December 20 and 21, 2006. (The state is in the shape of a rectangle that measures 388 mi west to east and 276 mi south to north.) Use the contour map to estimate the average snowfall for the entire state of Colorado on those days.



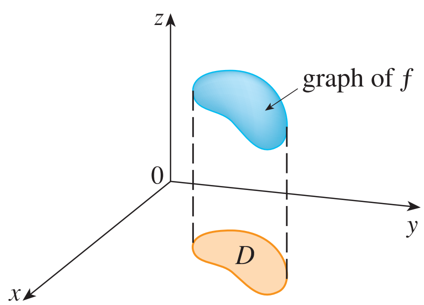
15.2 Double Integrals over General Regions

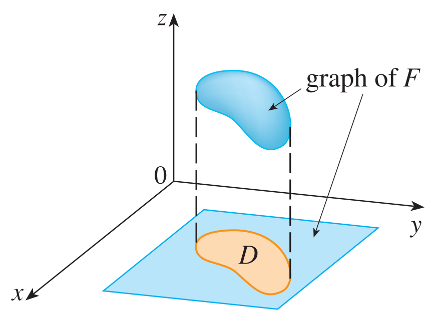
Definition 15.2.1. If F is integrable over R and D is a bounded region then we define the double integral of f over D by

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

where F is given by

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D, \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D. \end{cases}$$

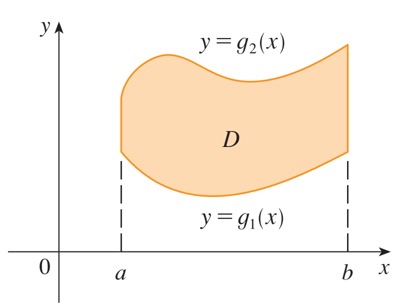


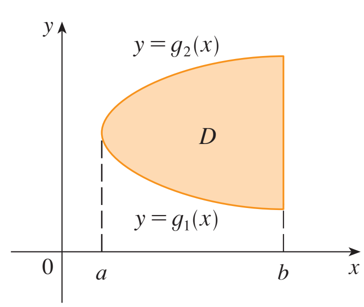


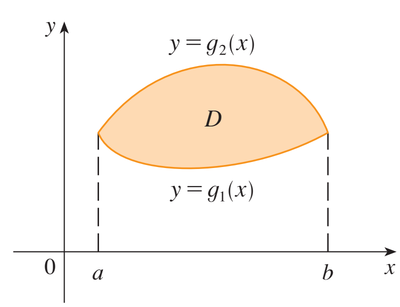
Definition 15.2.2. A plane region D is said to be of type I if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous on $[a,b]$. Some examples of type I regions are shown in the figure.







Theorem 15.2.1. *If f is continuous on a type I region D such that*

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

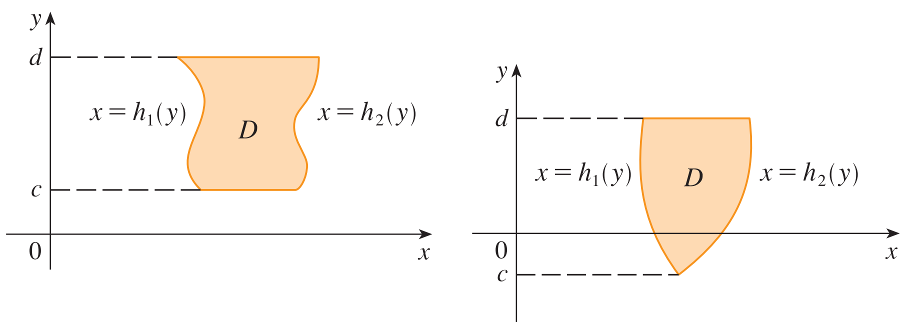
then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

Definition 15.2.3. A plane region D is said to be of type II if it lies between the graphs of two continuous functions of y, that is,

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

where h_1 and h_2 are continuous on $[c, d]$. Some examples of type II regions are shown in the figure.



Theorem 15.2.2. *If f is continuous on a type II region D such that*

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Example 1. Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Example 2. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Example 3. Evaluate $\iint_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Example 4. Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Example 5. Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

Theorem 15.2.3 (Properties of Double Integrals)**.**

1. $\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA.$

2. $\iint_D cf(x,y)\, dA = c \iint_D f(x,y)\, dA$ *where c is a constant.*

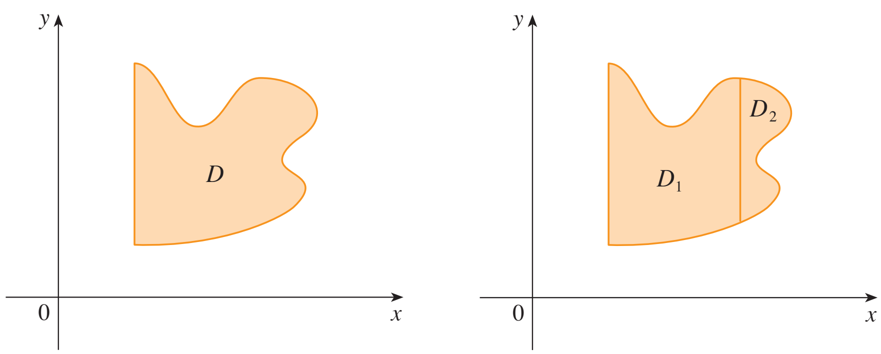
3. *If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then*

$$\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA.$$

4. *If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries, then*

$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$$

This property can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II, as illustrated by the figure.



(a) D is neither type I nor type II. (b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

5. $\iint_D 1\, dA = A(D)$ *where $A(D)$ is the area of D.*

6. *If $m \leq f(x,y) \leq M$ for all (x,y) in D, then*

$$mA(D) \leq \iint_D f(x,y)\, dA \leq MA(D).$$

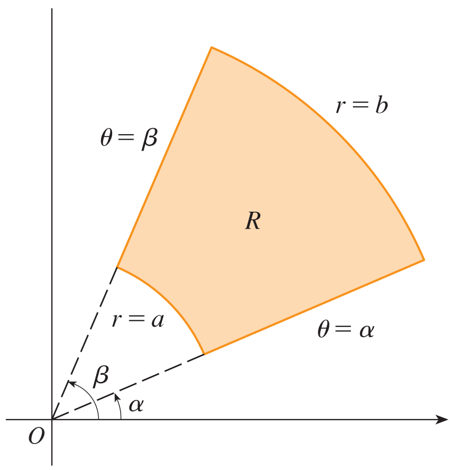
Example 6. Use Property 6 to estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and radius 2.

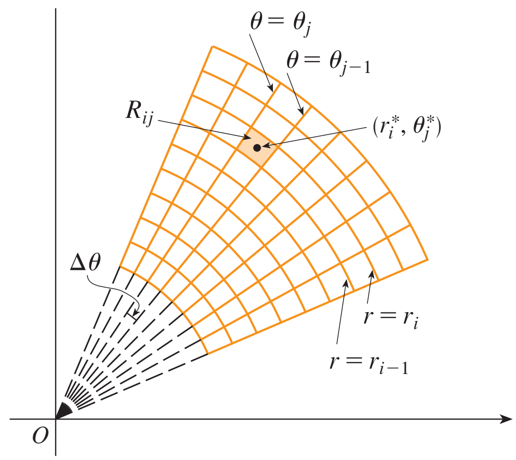
15.3 Double Integrals in Polar Coordinates

Definition 15.3.1. The region given by

$$R = \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

is called a polar rectangle, as shown in the figure.





Theorem 15.3.1 (Change to Polar Coordinates in a Double Integral)**.** *If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then*

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Proof. The "center" of the polar subrectangle

$$R_{ij} = \{(r, \theta) \mid r_{i-1} \leq r \leq r_i, \theta_{j-1} \leq \theta \leq \theta_j\}$$

has polar coordinates

$$r_i^* = \frac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j).$$

Since the area of a sector of a circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$, the area of R_{ij} is

$$\begin{aligned} \Delta A_i &= \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta \\ &= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^*\Delta r\Delta\theta. \end{aligned}$$

Therefore we have

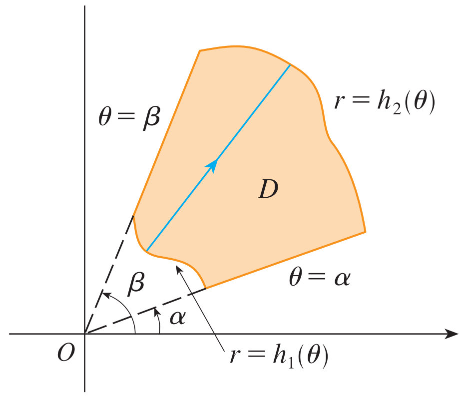
$$\begin{aligned}\iint_R f(x,y)\,dA &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\,\Delta A_i \\ &= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta. \qquad \square\end{aligned}$$

Example 1. Evaluate $\iint_R (3x+4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Example 2. Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z=1-x^2-y^2$.

Theorem 15.3.2. *If f is continuous on a polar region of the form*

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$



then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Example 3. Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Example 4. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

15.4 Applications of Double Integrals

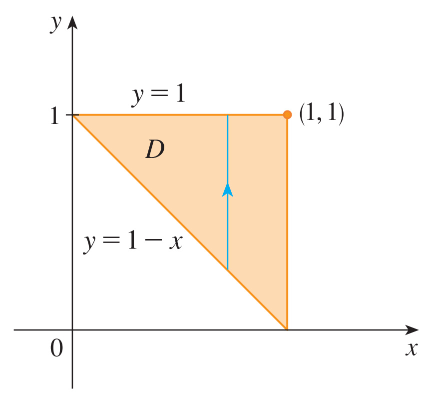
Definition 15.4.1. Suppose a lamina occupies a region D of the xy-plane and its density (in units of mass per unit area) at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D. Then the total mass of the lamina is given by

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D \rho(x, y)\, dA.$$

Similarly, if an electric charge is distributed over a region D and the charge density (in units of charge per unit area) is given by $\sigma(x, y)$ at a point (x, y) in D, then the total charge Q is given by

$$Q = \iint_D \sigma(x, y)\, dA.$$

Example 1. Charge is distributed over the triangular region D in the figure so that the charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.



Definition 15.4.2. Suppose a lamina occupies a region D and has density function $\rho(x, y)$. The moment of the lamina about the x-axis is

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D y\rho(x, y)\, dA.$$

Similarly, moment about the y-axis is

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D x\rho(x, y)\, dA.$$

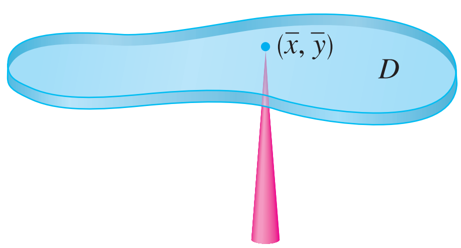
Definition 15.4.3. The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\rho(x, y)\, dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y\rho(x, y)\, dA$$

where the mass m is given by

$$m = \iint_D \rho(x, y)\, dA.$$

The lamina balances horizontally when supported at its center of mass (see the figure).



Example 2. Find the mass and center of mass of a triangular lamina with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$ if the density function is $\rho(x, y) = 1 + 3x + y$.

Example 3. The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

Definition 15.4.4. The moment of inertia (also called the second moment) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. The moment of inertia of the lamina about the x-axis is defined to be

$$I_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D y^2 \rho(x, y)\, dA.$$

Similarly, the moment of inertia about the y-axis is defined to be

$$I_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D x^2 \rho(x, y)\, dA.$$

The moment of inertia about the origin, also called the polar moment of inertia is defined to be

$$I_0 = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D (x^2 + y^2)\rho(x, y)\, dA.$$

Example 4. Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x, y) = \rho$, center the origin, and radius a.

Definition 15.4.5. The radius of gyration of a lamina about an axis is the number R such that

$$mR^2 = I$$

where m is the mass of the lamina and I is the moment of inertia about the given axis. In particular, the radius of gyration $\bar{\bar{y}}$ with respect to the x-axis and the radius of gyration $\bar{\bar{x}}$ with respect to the y-axis are given by the equations

$$m\bar{\bar{y}}^2 = I_x \qquad m\bar{\bar{x}}^2 = I_y.$$

Example 5. Find the radius of gyration about the x-axis of the disk in Example 4.

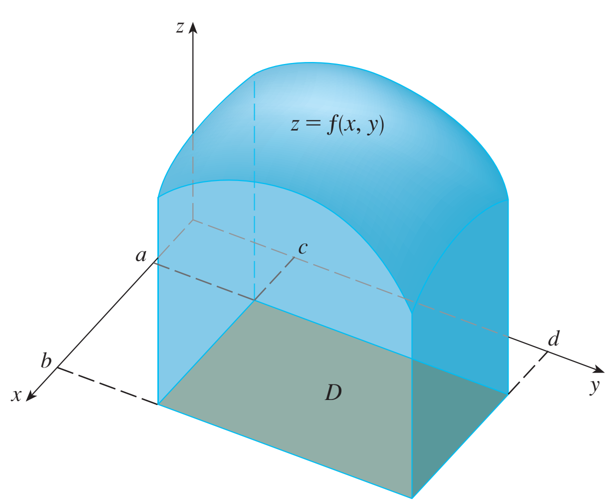
Definition 15.4.6. The joint density function of two continuous random variables X and Y is a function f of two variables such that the probability that (X, Y) lies in a region D is

$$P((X,Y) \in D) = \iint_D f(x,y)\, dA.$$

In particular, if the region is a rectangle, the probability that X lies between a and b and Y lies between c and d is

$$P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x,y)\, dy\, dx.$$

(See the figure.)



Remark 1. Because probabilities aren't negative and are measured on a scale from 0 to 1, the joint density function has the following properties:

$$f(x,y) \geq 0 \qquad \iint_{\mathbb{R}^2} f(x,y)\,dA = 1$$

for

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \lim_{a\to\infty}\iint_{D_a} f(x,y)\,dA$$

where D_a is the disk with radius a and center the origin.

Example 6. If the joint density function for X and Y is given by

$$f(x,y) = \begin{cases} C(x+2y) & \text{if } 0 \leq x \leq 10,\ 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

find the value of the constant C. Then find $P(X \leq 7, Y \geq 2)$.

Definition 15.4.7. Suppose X is a random variable with probability density function $f_1(x)$ and Y is a random variable with density function $f_2(y)$. Then X and Y are called independent random variables if their joint density function is the product of their individual density functions:

$$f(x,y) = f_1(x)f_2(y).$$

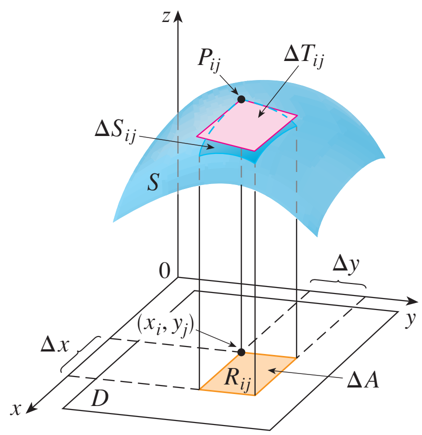
Example 7. The manager of a movie theater determines that the average time moviegoers wait in line to buy a ticket for this week's film is 10 minutes and the average time they wait to buy popcorn is 5 minutes. Assuming that the waiting times are independent, find the probability that a moviegoer waits a total of less than 20 minutes before taking his or her seat.

Definition 15.4.8. If X and Y are random variables with joint density function f, we define the X-mean and Y-mean, also called the expected values of X and Y, to be

$$\mu_1 = \iint_{\mathbb{R}^2} xf(x,y)\,dA \qquad \mu_2 = \iint_{\mathbb{R}^2} yf(x,y)\,dA.$$

Example 8. A factory produces (cylindrically shaped) roller bearings that are sold as having diameter 4.0 cm and length 6.0 cm. In fact, the diameters X are normally distributed with mean 4.0 cm and standard deviation 0.01 cm while the lengths Y are normally distributed with mean 6.0 cm and standard deviation 0.01 cm. Assuming that X and Y are independent, write the joint density function and graph it. Find the probability that a bearing randomly chosen from the production line has either length or diameter that differs from the mean by more than 0.02 cm.

15.5 Surface Area



Definition 15.5.1. Let S be a surface with equation $z = f(x, y)$, where f has continuous partial derivatives. We define the surface area of S to be

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

where ΔT_{ij} is the part of the tangent plane to S at the point P_{ij} on the surface corresponding to a rectangle R_{ij} in the domain D of f.

Theorem 15.5.1. *The area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are continuous, is*

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA.$$

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be the vectors that start at P_{ij} and lie along the sides of the parallelogram with area ΔT_{ij}. Then $\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|$. Since $f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ are the slopes of the tangent lines through P_{ij} in the directions of $\mathbf{a}$ and $\mathbf{b}$, we have

$$\mathbf{a} = \Delta x\,\mathbf{i} + f_x(x_i, y_j)\,\Delta x\,\mathbf{k}$$
$$\mathbf{b} = \Delta y\,\mathbf{j} + f_y(x_i, y_j)\,\Delta y\,\mathbf{k}.$$

and

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x(x_i, y_j)\,\Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\,\Delta y \end{vmatrix} \\ &= -f_x(x_i, y_j)\,\Delta x\,\Delta y\,\mathbf{i} - f_y(x_i, y_j)\,\Delta x\,\Delta y\,\mathbf{j} + \Delta x\,\Delta y\,\mathbf{k} \\ &= [-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}]\,\Delta A.\end{aligned}$$

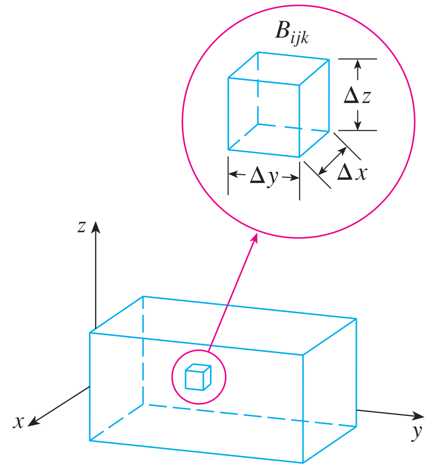
Thus

$$\begin{aligned}A(S) &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij} = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{a} \times \mathbf{b}| \\ &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\,\Delta A.\end{aligned}$$ □

Example 1. Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

Example 2. Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

15.6 Triple Integrals



Definition 15.6.1. The triple integral of f over the box B is

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta V$$

if this limit exists. The points $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ are called sample points, $\Delta V = \Delta x\,\Delta y\,\Delta z$ is the volume of the sub-box $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$, and the sum is called a triple Riemann sum.

Theorem 15.6.1 (Fubini's Theorem for Triple Integrals)**.** *If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$, then*

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

Example 1. Evaluate the triple integral $\iiint_R xyz^2\,dV$ where B is the rectangular box given by

$$B = \{(x,y,z) \mid 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}.$$

Definition 15.6.2. If F is integrable over B and E is a bounded region then we define the triple integral of f over E by

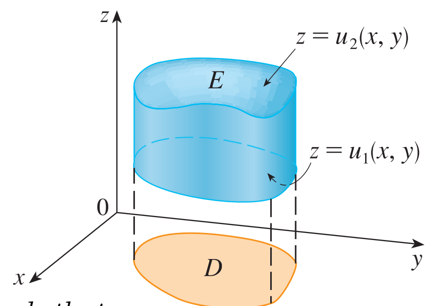
$$\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV$$

where F is defined so that it agrees with f on E but is 0 for points in B that are outside E.

Definition 15.6.3. A solid region E is said to be of type 1 if it lies between the graphs of two continuous functions of x and y, that is

$$E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$$

where D is the projection of E onto the xy-plane as shown in the figure.



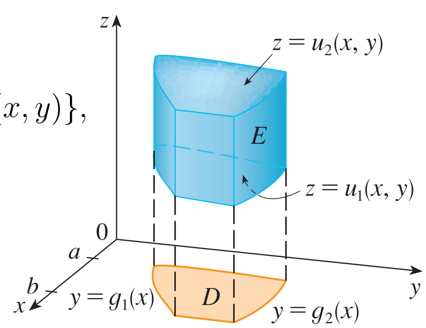
Theorem 15.6.2. *If f is continuous on a type 1 region E such that*

$$E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$$

then

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA.$$

Remark 1. If the projection D of E onto the xy-plane is a type I plane region (as in the figure), then

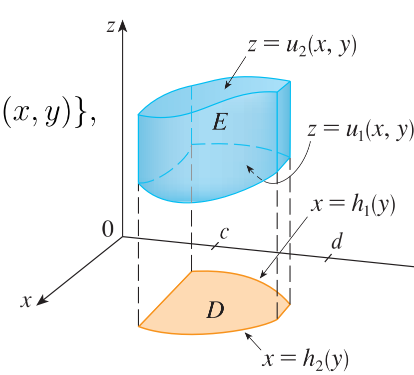


$$E = \{(x,y,z) \mid a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x,y) \le z \le u_2(x,y)\},$$

so

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

If, on the other hand, D is a type II plane region (as in the figure), then



$$E = \{(x,y,z) \mid c \le y \le d, h_1(y) \le x \le h_2(y), u_1(x,y) \le z \le u_2(x,y)\},$$

so

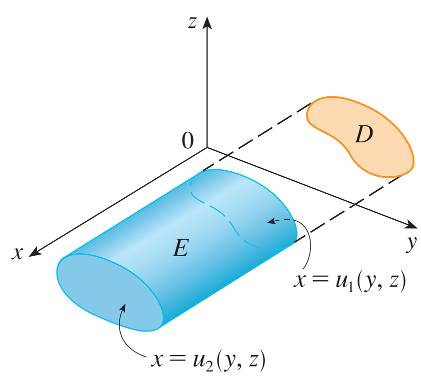
$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy.$$

Example 2. Evaluate $\iiint_E z\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Definition 15.6.4. A solid region E is of type 2 if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

where D is the projection of E onto the yz-plane as shown in the figure.



Theorem 15.6.3. *If f is continuous on a type 2 region E such that*

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

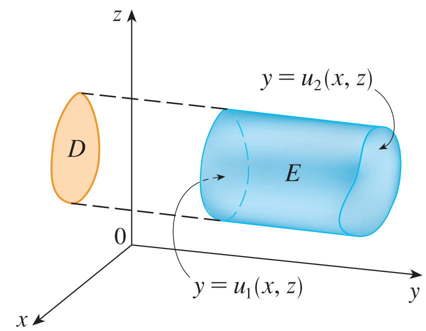
then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA.$$

Definition 15.6.5. A solid region E is of type 3 if it is of the form

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \le y \le u_2(x, z)\}$$

where D is the projection of E onto the xz-plane as shown in the figure.



Theorem 15.6.4. *If f is continuous on a type 3 region E such that*

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \le y \le u_2(x, z)\}$$

then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA.$$

Example 3. Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Example 4. Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x, then z, and then y.

Theorem 15.6.5.

$$V(E) = \iiint_E dV.$$

Example 5. Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Definition 15.6.6. If the density function of a solid object that occupies the region E is $\rho(x, y, z)$, in units of mass per unit volume, at any given point (x, y, z), then its mass is

$$m = \iiint_E \rho(x, y, z)\, dV$$

and its moments about the three coordinate planes are

$$M_{yz} = \iiint_E x\rho(x, y, z)\, dV \qquad M_{xz} = \iiint_E y\rho(x, y, z)\, dV$$
$$M_{xy} = \iiint_E z\rho(x, y, z)\, dV.$$

The center of mass is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}.$$

If the density is constant, the center of mass of the solid is called the centroid of E. The moments of inertia about the three coordinate axes are

$$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\, dV \qquad I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)\, dV$$
$$I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\, dV.$$

Definition 15.6.7. The total electric charge on a solid object occupying a region E and having charge density $\sigma(x, y, z)$ is

$$Q = \iiint_E \sigma(x, y, z)\, dV.$$

Definition 15.6.8. If we have three continuous random variables X, Y, and Z, their joint density function is a function of three variables such that the probability that (X, Y, Z) lies in E is

$$P((X, Y, Z) \in E) = \iiint_E f(x, y, z)\, dV.$$

In particular,

$$P(a \le X \le b, c \le Y \le d, r \le Z \le s) = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx.$$

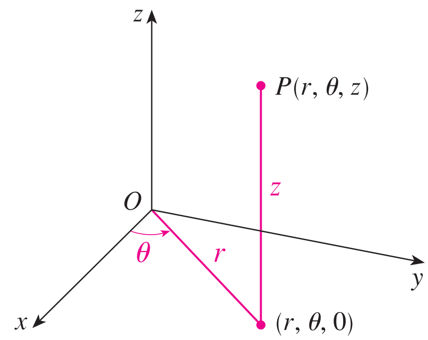
The joint density function satisfies

$$f(x, y, z) \ge 0 \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\, dz\, dy\, dx = 1.$$

Example 6. Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.

15.7 Integrals in Cylindrical Coordinates

Definition 15.7.1. In the cylindrical coordinate system, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P. (See the figure.)



Theorem 15.7.1. *To convert from cylindrical to rectangular coordinates, we use the equations*

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

whereas to convert from rectangular to cylindrical coordinates, we use

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z.$$

Example 1. (a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.

(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

Example 2. Describe the surface whose equation in cylindrical coordinates is $z = r$.

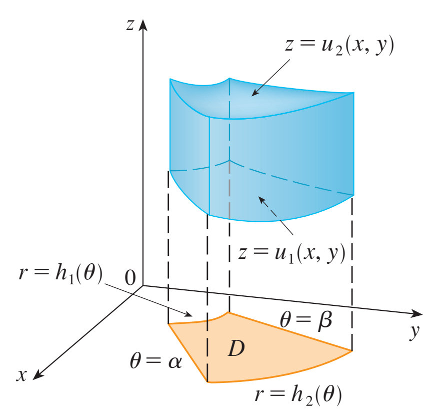
Theorem 15.7.2. *Suppose that E is a type 1 region whose projection D onto the xy-plane is described in polar coordinates (see the figure). In particular, suppose that f is continuous and*

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is given in polar coordinates by

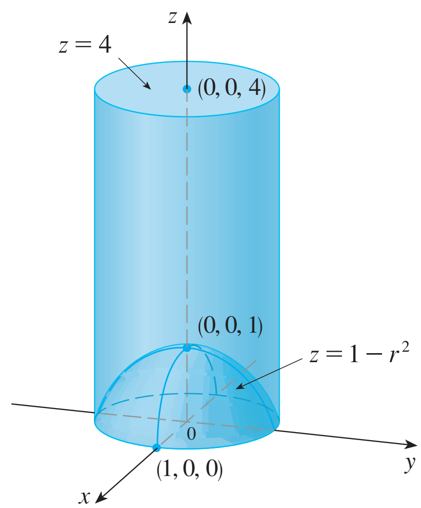
$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}.$$

Then the formula for triple integration in cylindrical coordinates is



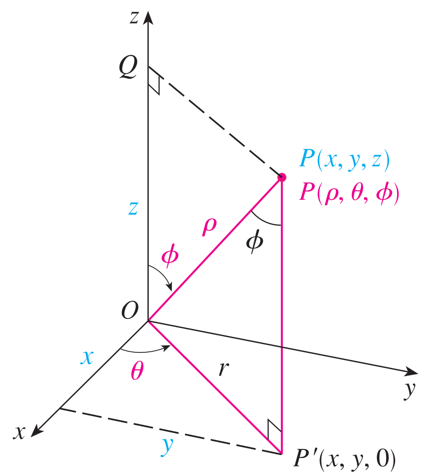
$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta.$$

Example 3. A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See the figure.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.



Example 4. Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2 + y^2)\,dz\,dy\,dx.$

15.8 Integrals in Spherical Coordinates



Definition 15.8.1. The spherical coordinates (ρ, θ, ϕ) of a point P in space are shown in the figure, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP. Note that

$$\rho \geq 0 \qquad 0 \leq \phi \leq \pi.$$

Theorem 15.8.1. *The relationship between rectangular and spherical coordinates can be seen from the figure. To convert from spherical to rectangular coordinates, we use the equations*

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi.$$

To convert from rectangular to spherical coordinates, we use the equation

$$\rho^2 = x^2 + y^2 + z^2.$$

Example 1. The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

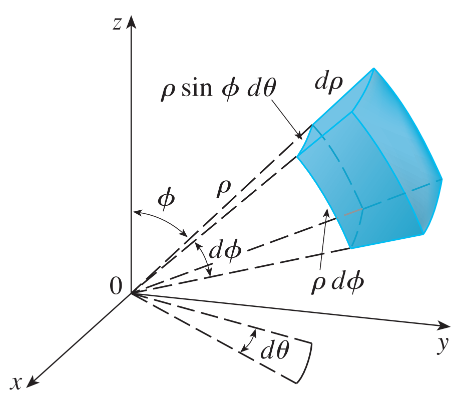
Example 2. The point $\left(0, 2\sqrt{3}, -2\right)$ is given in rectangular coordinates. Find spherical coordinates for this point.

Theorem 15.8.2. *The formula for triple integration in spherical coordinates is*

$$\iiint_E f(x,y,z)\,dV$$
$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

where E is a spherical wedge given by

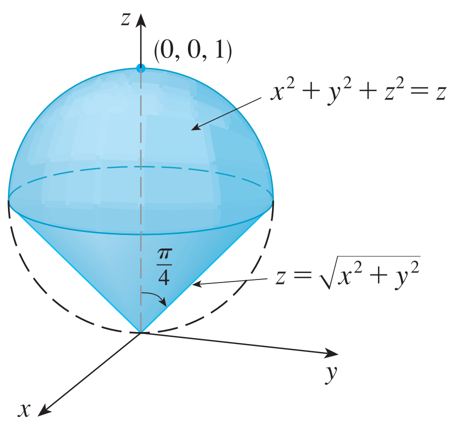
$$E = \{(\rho,\theta,\phi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}.$$



Example 3. Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}.$$

Example 4. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. (See the figure.)



15.9 Change of Variables in Multiple Integrals

Definition 15.9.1. A change of variables is given by a transformation T from the uv-plane to the xy-plane:

$$T(u, v) = (x, y)$$

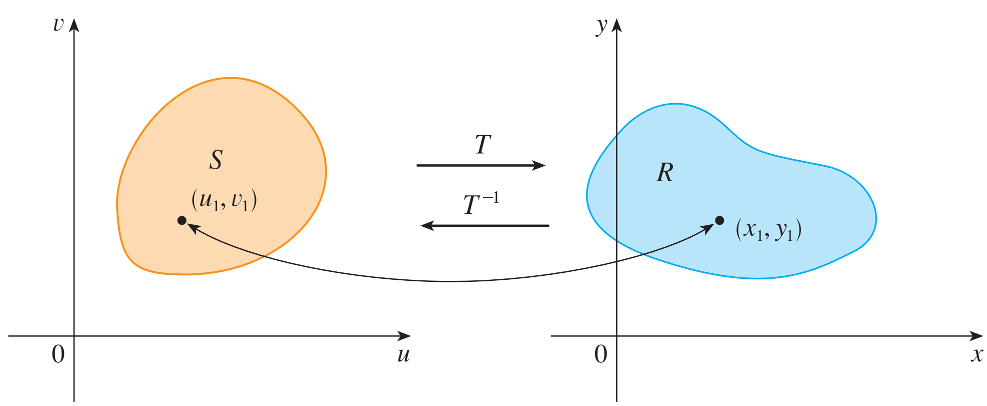
where x and y are related to u and v by the equations

$$x = g(u, v) \qquad y = h(u, v).$$

We usually assume that T is a C^1 transformation, which means that g and h have continuous first-order partial derivatives.

Remark 1. A transformation T is really just a function whose domain and range are both subsets of $\mathbb{R}^2$. If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the image of the point (u_1, v_1). If no two points have the same image, T is called one-to-one. The figure shows the effect of a transformation T on a region S in the uv-plane. T transforms S into a region R in the xy-plane called the image of S, consisting of the images of all points in S.

If T is a one-to-one transformation, then it has an inverse transformation



T^{-1} from the xy-plane to the uv-plane and it may be possible to solve for u and v in terms of x and y:

$$u = G(x, y) \qquad v = H(x, y).$$

Example 1. A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv.$$

Find the image of the square $S = \{(u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq 1\}$.

Definition 15.9.2. The Jacobian of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Remark 2. This notation can be used to show that the area ΔA of the image R in the xy-plane of a rectangle in the uv-plane is approximately

$$\Delta A \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u \, \Delta v.$$

Theorem 15.9.1 (Change of Variables in a Double Integral)**.** *Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then*

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv.$$

Example 2. Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y \, dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.

Example 3. Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\, dA$ where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$.

Definition 15.9.3. The Jacobian of the transformation T given by $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$ is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Theorem 15.9.2 (Change of Variables in a Triple Integral)**.** *Under hypotheses similar to those in Theorem 15.9.1,*

$$\iiint_R f(x, y, z)\, dV = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw.$$

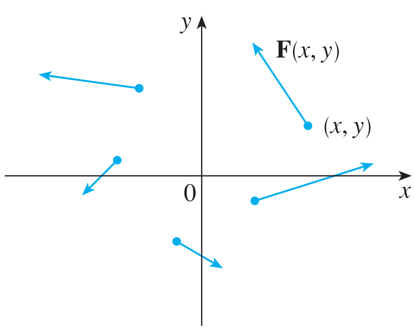
Example 4. Use Theorem 15.9.2 to derive the formula for triple integration in spherical coordinates.

Chapter 16

Vector Calculus

16.1 Vector Fields

Definition 16.1.1. Let D be a set in $\mathbb{R}^2$ (a plane region). A vector field on $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.



Remark 1. Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its component functions P and Q as follows:

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

or, for short,

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j}.$$

Note that P and Q are scalar functions of two variables and are sometimes called scalar fields to distinguish them from vector fields.

Definition 16.1.2. Let E be a subset of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

Remark 2. We can express a vector field $\mathbf{F}$ on $\mathbb{R}^3$ in terms of its component functions P, Q, and R as

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

Example 1. A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$. Describe $\mathbf{F}$ by sketching some of the vectors $\mathbf{F}(x, y)$.

Example 2. Sketch the vector field on $\mathbb{R}^3$ given by $\mathbf{F}(x, y, z) = z\mathbf{k}$.

Example 3. Imagine a fluid flowing steadily along a pipe and let $\mathbf{V}(x, y, z)$ be the velocity vector at a point (x, y, z). Then $\mathbf{V}$ assigns a vector to each point (x, y, z) in a certain domain E (the interior of the pipe) and so $\mathbf{V}$ is a vector field on $\mathbb{R}^3$ called a velocity field. Sketch a possible velocity field in a fluid flow.

Example 4. Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses m and M is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

where r is the distance between the objects and G is the gravitational constant. Let's assume that the object with mass M is located at the origin in $\mathbb{R}^3$ and let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z \rangle$. Write and sketch an equation for the gravitational field $\mathbf{F}$.

Example 5. Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the magnitude of the electric force $\mathbf{F}(\mathbf{x})$ exerted by this charge on a charge q located at a point (x, y, z) with position vector $\mathbf{x} = \langle x, y, z \rangle$ is

$$|\mathbf{F}| = \frac{\varepsilon q Q}{r^2}$$

where ε is a constant (that depends on the units used). This vector field and the one in Example 4 are examples of force fields. Instead of considering the electric force $\mathbf{F}$, physicists often consider the force per unit charge $\mathbf{E}(\mathbf{x}) = \frac{1}{q}\mathbf{F}(\mathbf{x})$, called the electric field of Q. Write equations for $\mathbf{F}$ and $\mathbf{E}$.

Definition 16.1.3. If f is a scalar function of two variables, its gradient

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

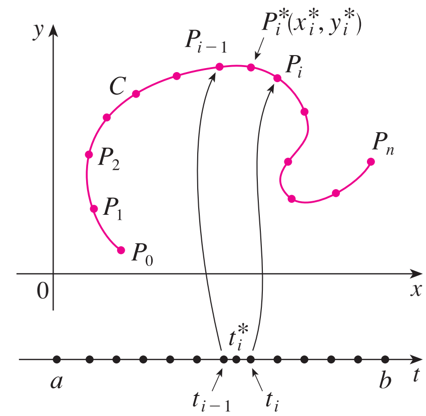
is a vector field on $\mathbb{R}^2$ called a gradient vector field. Likewise, if f is a scalar function of two variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Example 6. Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

Definition 16.1.4. A vector field $\mathbf{F}$ is called a conservative vector field if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a potential function for $\mathbf{F}$.

16.2 Line Integrals



Definition 16.2.1. If f is defined on a smooth curve C given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad a \leq t \leq b,$$

then the line integral of f along C is

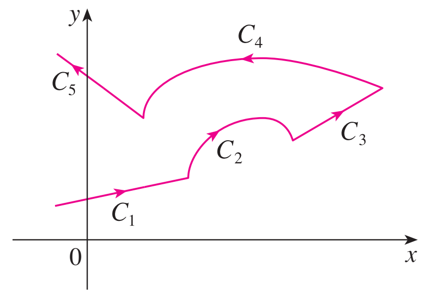
$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

if this limit exists. The lengths Δs_i are of subarcs of C and the points (x_i^*, y_i^*) are sample points in the ith subarc.

Remark 1. Using the formula for the length of C we can write

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example 1. Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.



Definition 16.2.2. Suppose that C is a piecewise-smooth curve; that is, C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where, as illustrated in the figure, the initial point of C_{i+1} is the terminal point of C_i. Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds.$$

Example 2. Evaluate $\int_C 2x\,ds$ where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

Definition 16.2.3. Suppose that $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve C. Then the mass m of the wire is given by

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*) \, \Delta s_i = \int_C \rho(x, y) \, ds.$$

The center of mass of the wire with density function ρ is located at the point $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{m} \int_C x\rho(x, y) \, ds \qquad \bar{y} = \frac{1}{m} \int_C y\rho(x, y) \, ds.$$

Example 3. A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geq 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

Definition 16.2.4. The integrals

$$\int_C f(x,y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta x_i$$
$$\int_C f(x,y)\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta y_i$$

are called the line integrals of f along C with respect to x and y. The original line integral $\int_C f(x,y)\,ds$ is called the line integral with respect to arc length.

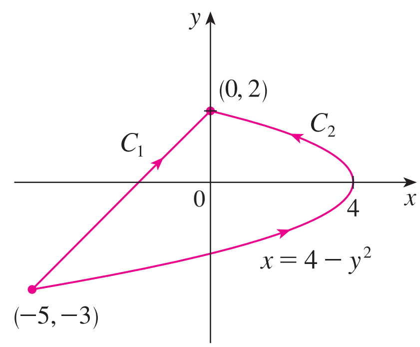
Theorem 16.2.1. *Line integrals with respect to x and y can also be evaluated by expressing everything in terms of t:*

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))x'(t)\,dt$$
$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))y'(t)\,dt.$$

Remark 2. When line integrals with respect to x and y occur together we abbreviate by writing

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)dx + Q(x,y)\,dy.$$

Example 4. Evaluate $\int_C y^2\,dx + x\,dy$, where (See the figure.)



(a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$.

Definition 16.2.5. Suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b,$$

or by a vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. If f is a function three variables that is continuous on some region containing C, then the line integral of f along C (with respect to arc length) is

$$\int_C f(x, y, z)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta s_i$$

if this limit exists.

Remark 3. Using the formula for the length of C we can write

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt,$$

or, more compactly,

$$\int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)|\, dt.$$

For the special case $f(x, y, z) = 1$, we get

$$\int_C ds = \int_a^b |\mathbf{r}'(t)|\, dt = L$$

where L is the length of the curve C.

Definition 16.2.6. The integrals

$$\int_C f(x, y, z)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta x_i = \int_a^b f(x(t), y(t), z(t))x'(t)\, dt$$

$$\int_C f(x, y, z)\, dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta y_i = \int_a^b f(x(t), y(t), z(t))y'(t)\, dt$$

$$\int_C f(x, y, z)\, dz = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta z_i = \int_a^b f(x(t), y(t), z(t))z'(t)\, dt$$

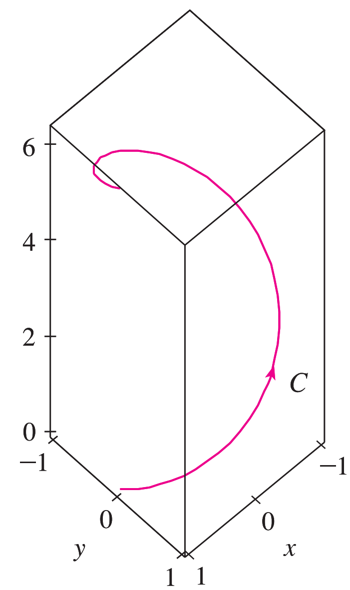
are called the line integrals of f along C with respect to x, y, and z.

Remark 4. As with line integrals in the plane, we evaluate integrals of the form

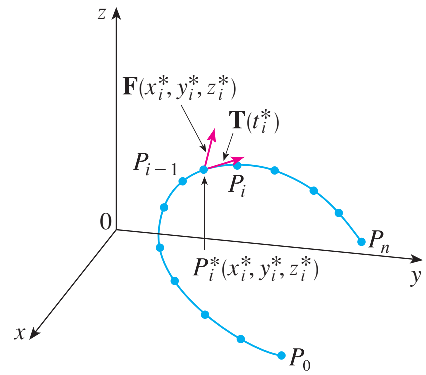
$$\int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$$

by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t.

Example 5. Evaluate $\int_C y \sin z\, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq 2\pi$. (See the figure.)



Example 6. Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C consists of the line segment C_1 from $(2, 0, 0)$ to $(3, 4, 5)$, followed by the vertical line segment C_2 from $(3, 4, 5)$ to $(3, 4, 0)$.



Definition 16.2.7. Suppose that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a continuous force field on $\mathbb{R}^3$. We define the work W done by the force field $\mathbf{F}$ as the limit of the Riemann sums

$$\sum_{i=1}^{n} [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)] \, \Delta s_i$$

where $P_i^*(x_i^*, y_i^*, z_i^*)$ is a point on the ith subarc $P_{i-1}P_i$ of C, and $\mathbf{T}(x, y, z)$ is the unit tangent vector at the point (x, y, z) on C. That is,

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) \, ds = \int_C \mathbf{F} \cdot \mathbf{T} \, ds.$$

Remark 5. If the curve C is given by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$, so

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)| \, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt,$$

which we abbreviate as $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Definition 16.2.8. Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \le t \le b$. Then the line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} \, ds.$$

Example 7. Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \le t \le \pi/2$.

Example 8. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leq t \leq 1.$$

Theorem 16.2.2. *Suppose the vector field* $\mathbf{F}$ *on* $\mathbb{R}^3$ *is given in component form by* $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$*. Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz.$$

Proof.

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_a^b (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k})\,dt \\
&= \int_a^b \big[P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)\big]\,dt
\end{aligned}$$

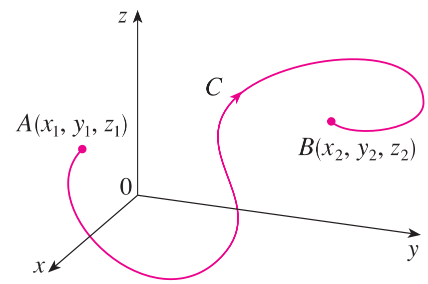
□

16.3 Fundamental Theorem for Line Integrals

Theorem 16.3.1 (Fundamental Theorem for Line Integrals)**.** *Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then*

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Proof. If f is a function of three variables and C is a space curve joining the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$, as in the figure, then the theorem becomes



$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1).$$

In this case (the case for two variables is similar),

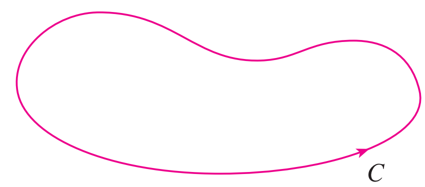
$$\begin{aligned}\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt \\ &= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \qquad \square\end{aligned}$$

Example 1. Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

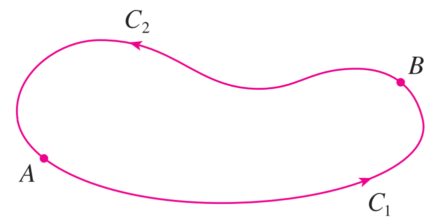
in moving a particle with mass m from the point $(3, 4, 12)$ to the point $(2, 2, 0)$ along a piecewise-smooth curve C. (See Example 16.1.4.)

Remark 1. In general, if $\mathbf{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D that have the same initial points and the same terminal points. By Theorem 16.3.1, line integrals of conservative vector fields are independent of path. A curve is called closed if its terminal point coincides with its initial point, that is, $\mathbf{r}(b) = \mathbf{r}(a)$. (See the figure.)



Theorem 16.3.2. *$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.*

Proof. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D and C is any closed path in D, we can choose any two points A and B on C as being composed of the path C_1 from A to B followed by the path C_2 from B to A. (See the figure.) Then



$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

since C_1 and $-C_2$ have the same initial and terminal points.
Conversely, if it is true that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ whenever C is a closed path in D, then we demonstrate independence of path as follows. Take any two paths C_1 and C_2 from A to B in D and define C to be the curve consisting of C_1 followed by $-C_2$. Then

$$0 = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

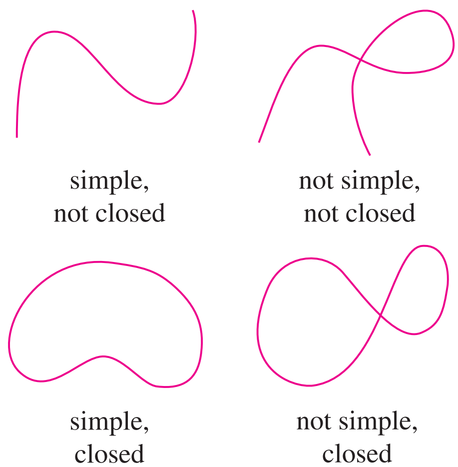
and so $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. □

Theorem 16.3.3. *Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. (By open we mean that for every point P in D there is a disk with center P that lies entirely in D, and by connected we mean that any two points in D can be joined by a path that lies in D.) If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.*

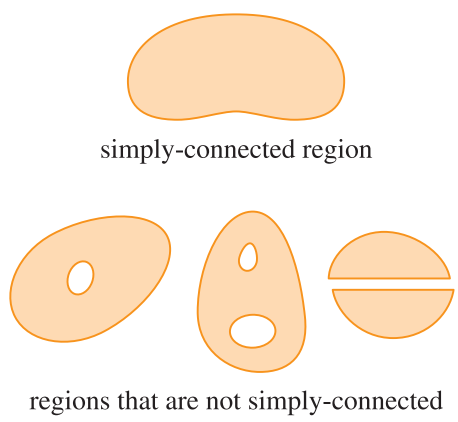
Theorem 16.3.4. *If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Definition 16.3.1. A simple curve is a curve that does not intersect itself anywhere between its endpoints. [See the figure; $\mathbf{r}(a) = \mathbf{r}(b)$ for a simple closed curve, but $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_2 < t_2 < b$.]



Definition 16.3.2. A simply-connected region in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D. [See the figure; a simply-connected region contains no hole and cannot consist of two separate pieces.]



Theorem 16.3.5. *Let* $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ *be a vector field on an open simply-connected region* D*. Suppose that* P *and* Q *have continuous first-order partial derivatives*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D.$$

Then $\mathbf{F}$ *is conservative.*

Example 2. Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x - 2)\mathbf{j}$$

is conservative.

Example 3. Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$$

is conservative.

Example 4. (a) If $\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

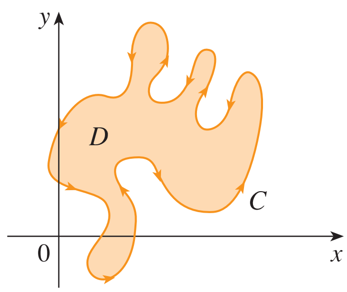
(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$$\mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j} \qquad 0 \leq t \leq \pi.$$

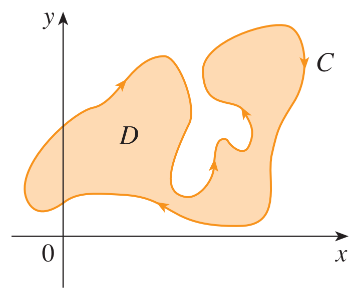
Example 5. If $\mathbf{F}(x, y, z) = y^2\mathbf{i} + (2xy + e^{3z})\mathbf{j} + 3ye^{3z}\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

16.4 Green's Theorem

Definition 16.4.1. The positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C. Thus if C is given by the vector function $\mathbf{r}(t)$, $a \le t \le b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C. (See the figure.)



(a) Positive orientation



(b) Negative orientation

Theorem 16.4.1 (Green's Theorem)**.** *Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then*

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Remark 1. The notation

$$\oint P\,dx + Q\,dy \qquad \text{or} \qquad \circlearrowleft\!\!\!\!\!\!\int P\,dx + Q\,dy$$

is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C. Another notation for the positively oriented boundary curve of D is ∂D, so the equation in Green's Theorem can be written as

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy.$$

Example 1. Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.

Example 2. Evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$, where C is the circle $x^2 + y^2 = 9$.

Theorem 16.4.2. *The area of a region D is*

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

Proof. Since the area of D is $\iint_D 1\,dA$, we wish to choose P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

There are several possibilities:

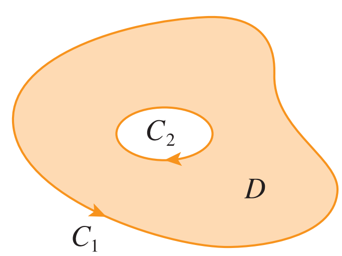
$$\begin{array}{lll} P(x,y) = 0 & P(x,y) = -y & P(x,y) = -\frac{1}{2}y \\ Q(x,y) = x & Q(x,y) = 0 & Q(x,y) = \frac{1}{2}x. \end{array}$$

Then the result follows by Green's Theorem. □

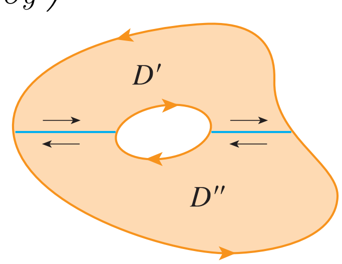
Example 3. Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Example 4. Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Remark 2. Green's Theorem can be extended to apply to regions with holes, that is, regions that are not simply-connected. Observe that the boundary C of the region D in the top figure consists of two simple closed curves C_1 and C_2. By dividing the region D into two regions D' and D'' by means of the lines shown in the bottom figure, and then applying Green's Theorem to each of D' and D'', we get



$$\begin{aligned}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\ &= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy \\ &= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy \\ &= \int_C P\,dx + Q\,dy\end{aligned}$$



Example 5. If $\mathbf{F}(x, y) = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

16.5 Curl and Divergence

Definition 16.5.1. If $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

Remark 1. The equation for curl can be rewritten using operator notation by introducing the vector differential operator ∇ ("del") as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

It has meaning when it operates on a scalar function to produce the gradient of f:

$$\nabla f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field $\mathbf{F}$ as follows:

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \\ &= \operatorname{curl}\mathbf{F}.\end{aligned}$$

Example 1. If $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$, find $\operatorname{curl}\mathbf{F}$.

Theorem 16.5.1. *If f is a function of three variables that has continuous second-order partial derivatives, then*

$$\operatorname{curl}(\nabla f) = \mathbf{0}.$$

Proof.

$$\begin{aligned}
\operatorname{curl}(\nabla f) = \nabla \times (\nabla f) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} \\
&= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k} \\
&= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}
\end{aligned}$$

by Clairaut's Theorem. □

Example 2. Show that the vector field $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$ is not conservative.

Theorem 16.5.2. *If* $\mathbf{F}$ *is a vector field defined on all of* $\mathbb{R}^3$ *whose component functions have continuous partial derivatives and* $\operatorname{curl} \mathbf{F} = \mathbf{0}$*, then* $\mathbf{F}$ *is a conservative vector field.*

Example 3. (a) Show that

$$\mathbf{F}(x, y, z) = y^2 z^3 \mathbf{i} + 2xyz^3 \mathbf{j} + 3xy^2 z^2 \mathbf{k}$$

is a conservative vector field.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

Definition 16.5.2. If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the divergence of $\mathbf{F}$ is the function of three variables defined by

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Remark 2. In terms of the gradient operator $\nabla = (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j} + (\partial/\partial z)\mathbf{k}$, the divergence of $\mathbf{F}$ can be written symbolically as the dot product of ∇ and $\mathbf{F}$:

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}.$$

Example 4. If $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$, find $\operatorname{div}\mathbf{F}$.

Theorem 16.5.3. *If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then*

$$\operatorname{div}\operatorname{curl}\mathbf{F} = 0.$$

Proof.

$$\begin{aligned}
\operatorname{div}\operatorname{curl}\mathbf{F} &= \nabla \cdot (\nabla \times \mathbf{F}) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} \\
&= 0.
\end{aligned}$$

□

Example 5. Show that the vector field $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$ can't be written as the curl of another vector field, that is, $\mathbf{F} \neq \operatorname{curl}\mathbf{G}$.

Theorem 16.5.4. *Suppose a plane region D, its boundary curve C, and the functions P and Q satisfy the hypotheses of Green's Theorem where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. Then*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA.$$

Proof. Regarding $\mathbf{F}$ as a vector field on $\mathbb{R}^3$ with third component 0, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy$$

and

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}.$$

Therefore

$$(\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y},$$

and the result follows by Green's Theorem. □

Theorem 16.5.5. *Suppose a plane region D, its boundary curve C, and the functions P and Q satisfy the hypotheses of Green's Theorem where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. Then*

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA.$$

Proof. If C is given by the vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad a \le t \le b$$

then the unit tangent vector is

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{j}$$

and the outward unit normal vector to C is given by

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{j}.$$

Thus

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathbf{n}\, ds &= \int_a^b (\mathbf{F} \cdot \mathbf{n})(t) |\mathbf{r}'(t)|\, dt \\
&= \int_a^b \left[\frac{P(x(t), y(t)) y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t), y(t)) x'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)|\, dt \\
&= \int_a^b P(x(t), y(t)) y'(t)\, dt - Q(x(t), y(t))\, x'(t)\, dt \\
&= \int_C P\, dy - Q\, dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA
\end{aligned}$$

by Green's Theorem. □

16.6 Parametric Surfaces and Their Areas

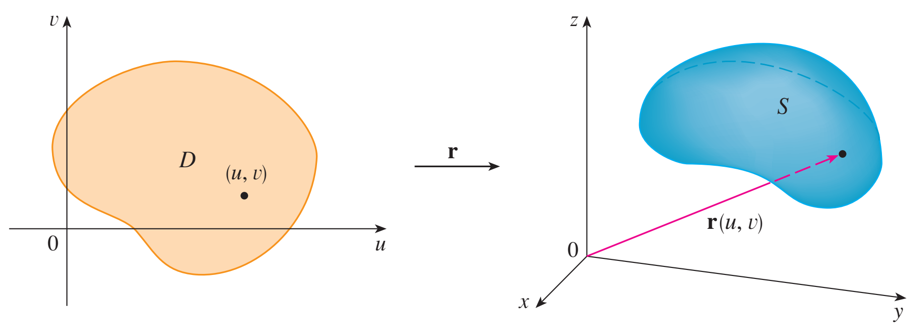
Definition 16.6.1. Suppose that

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

is a vector-valued function defined on a region D in the uv-plane. So x, y, and z, the component functions of $\mathbf{r}$, are functions of the two variables u and v with domain D. The set of all points (x, y, z) in $\mathbb{R}^3$ such that

$$x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v)$$

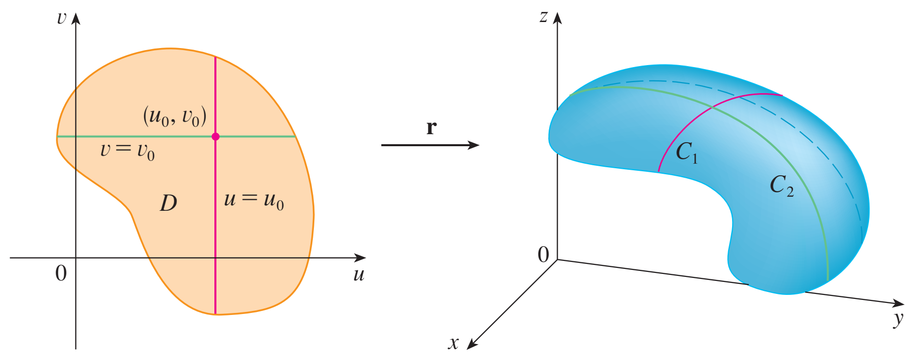
and (u, v) varies throughout D, is called a parametric surface S and the equations are called parametric equations of S. The surface S is traced out by the tip of the position vector $\mathbf{r}(u, v)$ as (u, v) moves throughout the region D. (See the figure.)



Example 1. Identify and sketch the surface with vector equation

$$\mathbf{r}(u,v) = 2\cos u\mathbf{i} + v\mathbf{j} + 2\sin u\mathbf{k}.$$

Definition 16.6.2. If a parametric surface S is given by a vector function $\mathbf{r}(u, v)$ and we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a curve C_1 lying on S. (See the figure.)



Similarly, if we keep v constant by putting $v = v_0$, we get a curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S. We call these curves grid curves.

Example 2. Use a computer algebra system to graph the surface

$$\mathbf{r}(u, v) = \langle (2 + \sin v) \cos u, (2 + \sin v) \sin u, u + \cos v \rangle.$$

Which grid curves have u constant? Which have v constant?

Example 3. Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and that contains two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$.

Example 4. Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2.$$

Example 5. Find a parametric representation for the cylinder

$$x^2 + y^2 = 4 \qquad 0 \leq z \leq 1.$$

Example 6. Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$.

Example 7. Find a parametric representation for the surface $z = 2\sqrt{x^2 + y^2}$, that is, the top half the cone $z^2 = 4x^2 + 4y^2$.

Example 8. Find parametric equations for the surface generated by rotating the curve $y = \sin x$, $0 \leq x \leq 2\pi$, about the x-axis. Use these equations to graph the surface of revolution.

Definition 16.6.3. If S is a parametric surface traced out by a vector function

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$, and if we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a grid curve C_1 lying on S. The tangent vector to C_1 at P_0 is obtained by taking the partial derivative of $\mathbf{r}$ with respect to v:

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$

Similarly, if we keep v constant by putting $v = v_0$, we get a grid curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S, and its tangent vector at P_0 is

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

If $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$, then the surface S is called smooth (it has no "corners"). For a smooth surface, the tangent plane is the plane that contains the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and the vector $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

Example 9. Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point $(1, 1, 3)$.

Definition 16.6.4. If a smooth parametric surface S is given by the equation

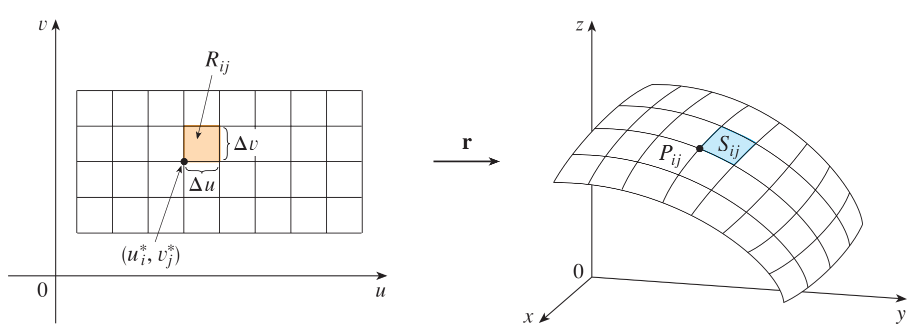
$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the surface area of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$$



Example 10. Find the surface area of a sphere of radius a.

Theorem 16.6.1. *If a surface S has equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives, then the surface areas of S is*

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

Proof. We take x and y as parameters. The parametric equations are

$$x = x \qquad y = y \qquad z = f(x, y)$$

so

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}.$$

Thus we have

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}. \qquad \square$$

Example 11. Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

16.7 Surface Integrals

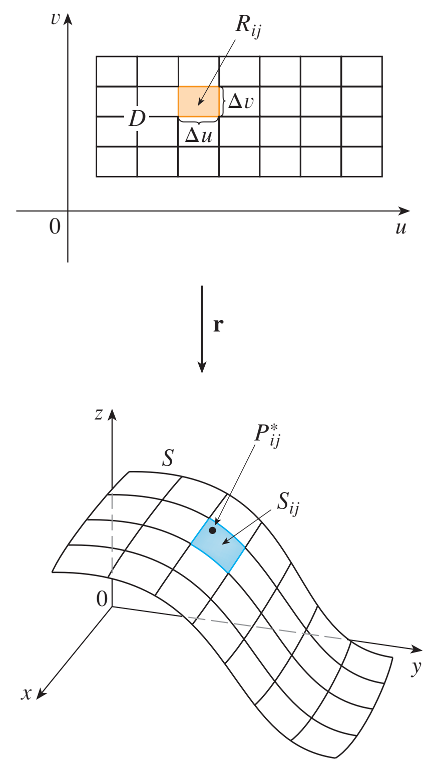
Definition 16.7.1. Suppose that a surface S has a vector equation

$$\mathbf{r}(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \qquad (u,v) \in D.$$

Then the <u>surface integral of f over the surface S</u> is

$$\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}$$

where the areas ΔS_{ij} are of patches of S that correspond to subrectangles R_{ij} with dimensions Δu and Δv, and the points P_{ij}^* are sample points in each patch.



Remark 1. It can be shown, even when the parameter domain D is not a rectangle, that

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA,$$

and thus

$$\iint_S 1\,dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA = A(S).$$

Example 1. Compute the surface integral $\iint_S x^2\,dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Theorem 16.7.1. *If S is a surface with equation $z = g(x, y)$, then*

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA.$$

Proof. Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x, y)$$

and so we have

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}.$$

Thus

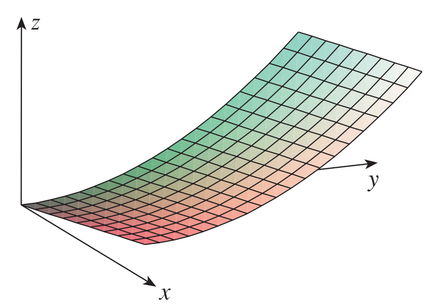
$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}.$$

□

Example 2. Evaluate $\iint_S y\, dS$, where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$. (See the figure.)

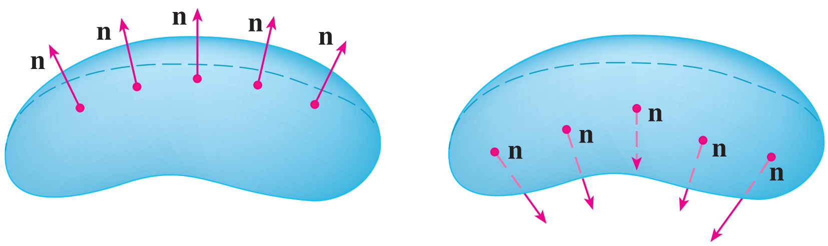


Definition 16.7.2. If S is a piecewise-smooth surface, that is, a finite union of smooth surfaces $S_1, S_2, \ldots, S_n$ that intersect only along their boundaries, then the surface integral of f over S is defined by

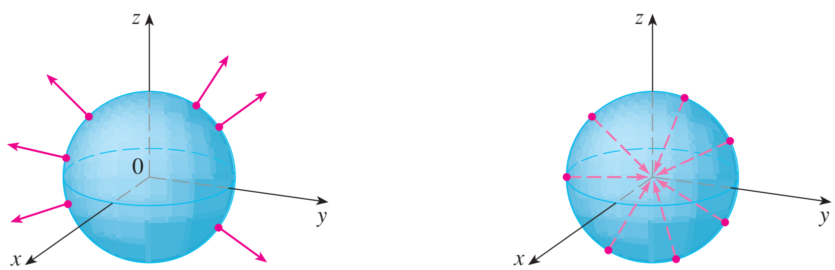
$$\iint_S f(x,y,z)\,dS = \iint_{S_1} f(x,y,z)\,dS + \cdots + \iint_{S_n} f(x,y,z)\,dS.$$

Example 3. Evaluate $\iint_S z\,dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \leq 1$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.

Definition 16.7.3. If S is a surface that has a tangent plane at every point (x, y, z) (except at any boundary point), and if it is possible to choose a unit normal vector $\mathbf{n}$ at every such point so that $\mathbf{n}$ varies continuously over S, then S is called an oriented surface and the given choice of $\mathbf{n}$ provides S with an orientation. There are two possible orientations for any orientable surface (see the figure).



Remark 2. For a closed surface, that is, a surface that is the boundary of a solid region E, the convention is that the positive orientation is the one for which the normal vectors point outward from E, and inward-pointing normals give the negative orientation (see the figure).



Positive orientation　　　　Negative Orientation

Definition 16.7.4. If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

This integral is also called the flux of $\mathbf{F}$ across S.

Theorem 16.7.2. *If S is given by a vector function $\mathbf{r}(u, v)$, then*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA$$

where D is the parameter domain.

Proof. If S is given by a vector function $\mathbf{r}(u,v)$, then $\mathbf{n}$ is given by

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

and thus we have

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, dS \\ &= \iint_D \left[\mathbf{F}(\mathbf{r}(u,v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\right] |\mathbf{r}_u \times \mathbf{r}_v|\, dA. \qquad \square\end{aligned}$$

Example 4. Find the flux of the vector field $\mathbf{F}(x,y,z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Remark 3. In the case of a surface S given by a graph $z = g(x, y)$, we can think of x and y as parameters and write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \right).$$

Thus

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA.$$

Example 5. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Definition 16.7.5. If $\mathbf{E}$ is an electric field, then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the electric flux of $\mathbf{E}$ through the surface S. Gauss's Law says that the net charge enclosed by a closed surface S is

$$Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

where ε_0 is a constant (called the permittivity of free space) that depends on the units used.

Definition 16.7.6. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then the heat flow is defined as the vector field

$$\mathbf{F} = -K\nabla u$$

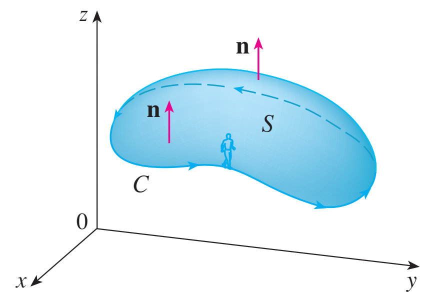
where K is an experimentally determined constant called the conductivity of the substance. The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}.$$

Example 6. The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

16.8 Stokes' Theorem

Definition 16.8.1. The figure shows an oriented surface with unit normal vector $\mathbf{n}$. The orientation of S induces the positive orientation of the boundary curve C shown in the figure. This means that if you walk in the positive direction around C with your head pointing in the direction of $\mathbf{n}$, then the surface will always be on your left.

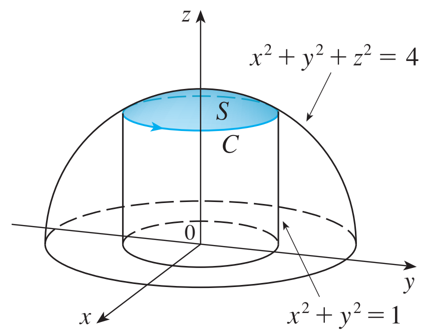


Theorem 16.8.1 (Stokes' Theorem)**.** *Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

Example 1. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above).

Example 2. Use Stokes' Theorem to compute the integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2+y^2 = 1$ and above the xy-plane. (See the figure.)



16.9 The Divergence Theorem

Definition 16.9.1. Regions E that are simultaneously of types 1, 2, and 3 are called simple solid regions. The boundary of E is a closed surface, and we use the convention that the positive orientation is outward; that is, the unit normal vector $\mathbf{n}$ is directed outward from E.

Theorem 16.9.1 (The Divergence Theorem)**.** *Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then*

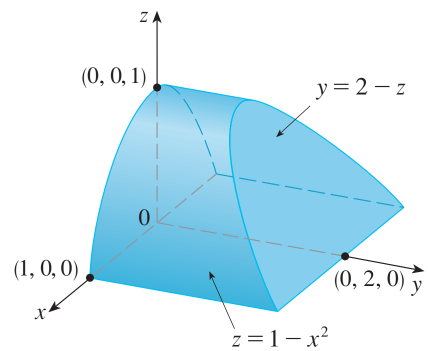
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.$$

Example 1. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Example 2. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + (y^2 + e^{xz^2})\mathbf{j} + \sin(xy)\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$. (See the figure.)



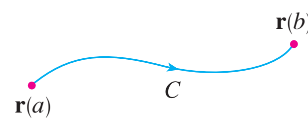
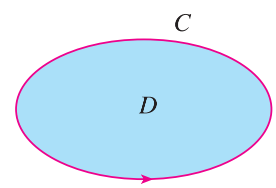
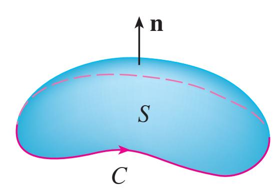
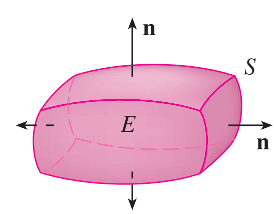
Example 3. In Example 16.1.5 we considered the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$$

where the electric charge Q is located at the origin and $\mathbf{x} = \langle x, y, z \rangle$ is a position vector. Use the Divergence Theorem to show that the electric flux of $\mathbf{E}$ through any closed surface S_2 that encloses the origin is

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = 4\pi\varepsilon Q.$$

16.10 Summary

Fundamental Theorem of Calculus	$\int_a^b F'(x)\,dx = F(b) - F(a)$	
Fundamental Theorem for Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$	
Green's Theorem	$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$	
Stokes' Theorem	$\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$	
Divergence Theorem	$\iiint_E \operatorname{div}\mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$	

Chapter 17

Second-Order Differential Equations

17.1 Second-Order Linear Equations

Definition 17.1.1. A second-order linear differential equation has the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

where P, Q, R, and G are continuous functions.

Definition 17.1.2. When $G(x) = 0$, for all x, in the equation in Definition 17.1.1. it is called a homogeneous linear equation. Thus the form of a second-order linear homogeneous differential equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0.$$

If $G(x) \neq 0$ for some x, the equation is nonhomogeneous.

Theorem 17.1.1. *If $y_1(x)$ and $y_2(x)$ are both solutions of a linear homogeneous equation and c_1 and c_2 are any constants, then the linear combination $y(x) = c_1y_1(x) + c_2y_2(x)$ is also a solution.*

Proof. Since y_1 and y_2 are solutions of a linear homogeneous equation, we have

$$P(x)y_1'' + Q(x)y_1' + R(x)y_1 = 0$$
$$P(x)y_2'' + Q(x)y_2' + R(x)y_2 = 0.$$

Therefore, using the basic rules for differentiation, we have

$$\begin{aligned}P(x)y''+&Q(x)y' + R(x)y \\ &= P(x)(c_1y_1 + c_2y_2)'' + Q(x)(c_1y_1 + c_2y_2)' + R(x)(c_1y_1 + c_2y_2) \\ &= P(x)(c_1y_1'' + c_2y_2'') + Q(x)(c_1y_1' + c_2y_2') + R(x)(c_1y_1 + c_2y_2) \\ &= c_1[P(x)y_1'' + Q(x)y_1' + R(x)y_1] + c_2[P(x)y_2'' + Q(x)y_2' + R(x)y_2] \\ &= c_1(0) + c_2(0) = 0.\end{aligned}$$

□

Definition 17.1.3. Solutions y_1 and y_2 to a linear homogeneous equation are linearly independent if neither y_1 nor y_2 is a constant multiple of the other. Otherwise, they are linearly dependent.

Theorem 17.1.2. *If y_1 and y_2 are linearly independent solutions of a linear homogeneous equation on an interval, and $P(x)$ is never 0, then the general solution is given by*

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

where c_1 and c_2 are arbitrary constants.

Remark 1. If $y = e^{rx}$ then $y' = re^{rx}$ and $y'' = r^2e^{rx}$, so $y = e^{rx}$ is a solution of $ay'' + by' + cy = 0$ if

$$\begin{aligned}ar^2e^{rx} + bre^{rx} + ce^{rx} &= 0 \\ (ar^2 + br + c)e^{rx} &= 0.\end{aligned}$$

But e^{rx} is never 0. Thus $y = e^{rx}$ is a solution if r is a root of the equation $ar^2 + br + c = 0$, called the auxiliary equation (or characteristic equation) of the differential equation $ay''+by'+cy = 0$. The roots r_1 and r_2 of the auxiliary equation can be found by factoring or using the quadratic formula:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Theorem 17.1.3 (Case I: $b^2 - 4ac > 0$)**.** *If the roots r_1 and r_2 of the auxiliary equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of $ay'' + by' + cy = 0$ is*

$$y = c_1e^{r_1x} + c_2e^{r_2x}.$$

Proof. In this case the roots r_1 and r_2 of the auxiliary equation are real and distinct, so $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are two linearly independent solutions of $ay'' + by' + cy = 0$. □

Example 1. Solve the equation $y'' + y' - 6y = 0$.

Example 2. Solve $3\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$.

Theorem 17.1.4 (Case II: $b^2 - 4ac = 0$)**.** *If the auxiliary equation $ar^2 + br + c = 0$ has only one real root r, then the general solution of $ay'' + by' + cy = 0$ is*

$$y = c_1 e^{rx} + c_2 x e^{rx}.$$

Proof. By the quadratic formula,

$$r = -\frac{b}{2a} \qquad \text{so} \quad 2ar + b = 0.$$

We know that $y_1 = e^{rx}$ is one solution of $ay'' + by' + cy = 0$. We now verify that $y_2 = xe^{rx}$ is also a solution:

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2re^{rx} + r^2 xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx} \\ &= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx} \\ &= 0(e^{rx}) + 0(xe^{rx}) = 0. \end{aligned}$$

Since $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent solutions, Theorem 17.1.2 provides us with the general solution. □

Example 3. Solve the equation $4y'' + 12y' + 9y = 0$.

Theorem 17.1.5 (Case III: $b^2 - 4ac < 0$)**.** *If the roots of the auxiliary equation* $ar^2 + br + c = 0$ *are the complex numbers* $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, *then the general solution of* $ay'' + by' + cy = 0$ *is*

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x).$$

Proof. Using Euler's equation

$$e^{i\theta} = \cos\theta + i \sin\theta,$$

we write the solution of the differential equation as

$$\begin{aligned} y &= C_1 e^{r_1 x} + C_2 e^{r_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} \\ &= C_1 e^{\alpha x}(\cos \beta x + i \sin \beta x) + C_2 e^{\alpha x}(\cos \beta x - i \sin \beta x) \\ &= e^{\alpha x}[(C_1 + C_2)\cos \beta x + i(C_1 - C_2)\sin \beta x] \\ &= e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \end{aligned}$$

where $c_1 = C_1 + C_2$, $c_2 = i(C_1 - C_2)$. □

Example 4. Solve the equation $y'' - 6y' + 13y = 0$.

Definition 17.1.4. An initial-value problem for a second-order linear differential equation consists of finding a solution y of the differential equation that also satisfies initial conditions of the form

$$y(x_0) = y_0 \qquad y'(x_0) = y_1$$

where y_0 and y_1 are given constants.

Example 5. Solve the initial-value problem

$$y'' + y' - 6y = 0 \qquad y(0) = 1 \qquad y'(0) = 0.$$

Example 6. Solve the initial-value problem

$$y'' + y = 0 \qquad y(0) = 2 \qquad y'(0) = 3.$$

Definition 17.1.5. A <u>boundary-value problem</u> for a second-order linear differential equation consists of finding a solution y of the differential equation that also satisfies boundary conditions of the form

$$y(x_0) = y_0 \qquad y(x_1) = y_1.$$

Example 7. Solve the boundary problem

$$y'' + 2y' + y = 0 \qquad y(0) = 1 \qquad y(1) = 3.$$

17.2 Nonhomogeneous Linear Equations

Theorem 17.2.1. *The general solution of the nonhomogeneous differential equation can be written as*

$$y(x) = y_p(x) + y_c(x)$$

where y_p is a particular solution of $ay'' + by' + cy = G(x)$ and y_c is the general solution of the <u>*complementary equation*</u> *$ay'' + by' + cy = 0$.*

Proof. We verify that if y is any solution of $ay'' + by' + cy = G(x)$, then $y - y_p$ is a solution of the complementary equation. Indeed

$$\begin{aligned} a(y-y_p)'' + b(y-y_p)' + c(y-y_p) &= ay'' - ay_p'' + by' - by_p' + cy - cy_p \\ &= (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p) \\ &= G(x) - G(x) = 0 \end{aligned}$$

This shows that every solution is of the form $y(x) = y_p(x) + y_c(x)$. It remains to show that every function of this form is a solution. Indeed

$$\begin{aligned} a(y_p+y_c)'' + b(y_p+y_c)' + c(y_p+y_c) &= ay_p'' + ay_c'' + by_p' + by_c' + cy_p + cy_c \\ &= (ay_p'' + by_p' + cy_p) + (ay_c'' + by_c' + cy_c) \\ &= G(x) + 0 = G(x). \end{aligned}$$

□

Remark 1 (The Method of Undetermined Coefficients).

1. If $G(x) = e^{kx}P(x)$, where P is a polynomial of degree n, then try $y_p(x) = e^{kx}Q(x)$, where $Q(x)$ is an nth-degree polynomial (whose coefficients are determined by substituting in the differential equation).

2. If $G(x) = e^{kx}P(x)\cos mx$ or $G(x) = e^{kx}P(x)\sin mx$, where P is an nth-degree polynomial, then try

   $$y_p(x) = e^{kx}Q(x)\cos mx + e^{kx}R(x)\sin mx$$

   where Q and R are nth-degree polynomials.

Modification: If any term of y_p is a solution of the complementary equation, multiply y_p by x (or by x^2 if necessary).

Example 1. Solve the equation $y'' + y' - 2y = x^2$.

Example 2. Solve $y'' + 4y = e^{3x}$.

Example 3. Solve $y'' + y' - 2y = \sin x$.

Example 4. Solve $y'' - 4y = xe^x + \cos 2x$.

Example 5. Solve $y'' + y = \sin x$.

Example 6. Determine the form of the trial solution for the differential equation $y'' - 4y' + 13y = e^{2x}\cos 3x$.

Remark 2. Suppose we have already solved the homogeneous equation $ay'' + by' + cy = 0$ and written the solution as

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where y_1 and y_2 are linearly independent solutions. We replace the constants (or parameters) c_1 and c_2 by arbitrary functions $u_1(x)$ and $u_2(x)$. We then look for a particular solution of the nonhomogeneous equation $ay'' + by' + cy = G(x)$ of the form

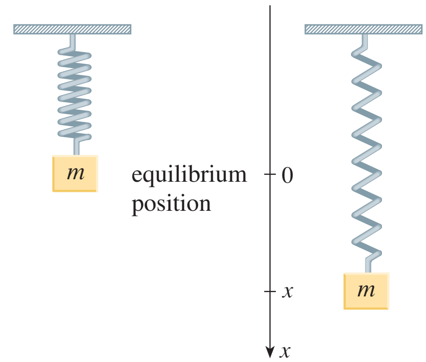
$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

This method is called variation of parameters because we have varied the parameters c_1 and c_2 to make them functions.

Example 7. Solve the equation $y'' + y = \tan x$, $0 < x < \pi/2$.

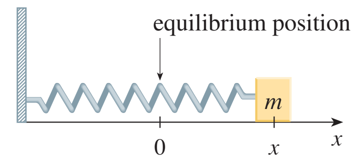
17.3 Applications of Differential Equations

Remark 1. Consider the motion of an object with mass m at the end of a spring that is either vertical (as in the first figure) or horizontal on a level surface (as in the second figure). Hooke's Law, which says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x:



$$\text{restoring force} = -kx$$

where k is a positive constant (called the spring constant). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have



$$m\frac{d^2x}{dt^2} = -kx \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + kx = 0.$$

This is a second-order linear differential equation. Its auxiliary equation is $mr^2 + k = 0$ with roots $r = \pm\omega i$, where $\omega = \sqrt{k/m}$. Thus the general solution is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

which can also be written as

$$x(t) = A \cos(\omega t + \delta)$$

where

$$\omega = \sqrt{k/m}$$
$$A = \sqrt{c_1^2 + c_2^2}$$
$$\cos \delta = \frac{c_1}{A} \qquad \sin \delta = -\frac{c_2}{A}.$$

This type of motion is called simple harmonic motion.

Example 1. A spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7 m. If the spring is stretched to a length of 0.7 m and then released with initial velocity 0, find the position of the mass at any time t.

Remark 2. Assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion. Thus

$$\text{damping force} = -c\frac{dx}{dt}$$

where c is a positive constant, called the damping constant. Thus, in this case, Newton's Second Law gives

$$m\frac{d^2x}{dt^2} = \text{restoring force} + \text{damping force} = -kx - c\frac{dx}{dt}$$

or

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

This is a second-order linear differential equation and its auxiliary equation is $mr^2 + cr + k = 0$. The roots are

$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \qquad r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}.$$

Case I: $c^2 - 4mk > 0$ (overdamping).
In this case r_1 and r_2 are distinct real roots and

$$x = c_1e^{r_1t} + c_2e^{r_2t}.$$

Case II: $c^2 - 4mk = 0$ (critical damping).
This case corresponds to equal roots

$$r_1 = r_2 = -\frac{c}{2m}$$

and the solution is given by

$$x = (c_1 + c_2t)e^{-(c/2m)t}.$$

Case III: $c^2 - 4mk < 0$ (underdamping).
Here the roots are complex:

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = -\frac{c}{2m} \pm \omega i$$

where

$$\omega = \frac{\sqrt{4mk - c^2}}{2m}.$$

The solution is given by

$$x = e^{-(c/2m)t}(c_1\cos\omega t + c_2\sin\omega t).$$

Example 2. Suppose that the spring of Example 1 is immersed in a fluid with damping constant $c = 40$. Find the position of the mass at any time t if it starts from the equilibrium position and is given a push to start it with an initial velocity of 0.6 m/s.

Remark 3. Suppose that, in addition to the restoring force and the damping force, the motion of the spring is affected by an external force $F(t)$. Then Newton's Second Law gives

$$\begin{aligned} m\frac{d^2x}{dt^2} &= \text{restoring force} + \text{damping force} + \text{external force} \\ &= -kx - c\frac{dx}{dt} + F(t). \end{aligned}$$

Thus, instead of the homogeneous equation, the motion of the spring is now governed by the following nonhomogeneous differential equation:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t).$$

Remark 4. The circuit shown in the figure contains an electromotive force E (supplied by a battery or generator), a resistor R, an indicator L, and a capacitor C, in series. If the charge on the capacitor at time t is $Q = Q(t)$, then the current is the rate of change of Q with respect to t: $I = dQ/dt$. It is known from physics that the voltage drops across the resistor, inductor and capacitor are

$$RI \qquad L\frac{dI}{dt} \qquad \frac{Q}{C}$$

respectively. Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage:

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t).$$

Since $I = dQ/dt$, this equation becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

which is a second-order linear differential equation with constant coefficients. If the charge Q_0 and the current I_0 are known at time 0, then we have the initial conditions

$$Q(0) = Q_0 \qquad Q'(0) = I(0) = I_0.$$

17.4 Series Solutions

Example 1. Use power series to solve the equation $y'' + y = 0$.

Example 2. Solve $y'' - 2xy' + y = 0$.

Index

Bibliography

[1] Stewart, James. *Calculus: Early Transcendentals.* Boston, MA, USA: Cengage Learning, 2016. Print.