# Multivariable Calculus Homework #5

Replace this text with your name

Due: Replace this text with a due date

**Exercise** (16.1.19)**.** If you have a CAS that plots vector fields (the command is `fieldplot` in Maple and `PlotVectorField` or `VectorPlot` in Mathematica), use it to plot

$$\mathbf{F}(x, y) = (y^2 - 2xy)\mathbf{i} + (3xy - 6x^2)\mathbf{j}.$$

Explain the appearance by finding the set of points $(x, y)$ such that $\mathbf{F}(x, y) = \mathbf{0}$.

*Solution*: Replace this text with your solution. □

**Exercise** (16.1.24)**.** Find the gradient vector field of $f(x, y, z) = x^2 y e^{y/z}$.

*Solution*: Replace this text with your solution. □

**Exercise** (16.1.34)**.** At time $t = 1$, a particle is located at position $(1, 3)$. If it moves in a velocity field

$$\mathbf{F}(x, y) = \langle xy - 2, y^2 - 10 \rangle$$

find its approximate location at time $t = 1.05$.

*Solution*: Replace this text with your solution. □

**Exercise** (16.2.34)**.** A thin wire has the shape of the first-quadrant part of the circle with center the origin and radius $a$. If the density function is $\rho(x, y) = kxy$, find the mass and center of mass of the wire.

*Solution*: Replace this text with your solution. ☐

**Exercise** (16.2.41)**.** Find the work done by the force field

$$\mathbf{F}(x, y, z) = \langle x - y^2, y - z^2, z - x^2 \rangle$$

on a particle that moves along the line segment from $(0, 0, 1)$ to $(2, 1, 0)$.

*Solution*: Replace this text with your solution. ☐

**Exercise** (16.2.45)**.** A 160-lb man carries a 25-lb can of paint up a helical staircase that encircles a silo with a radius of 20 ft. If the silo is 90 ft high and the man makes exactly three complete revolutions climbing to the top, how much work is done by the man against gravity?

*Solution*: Replace this text with your solution. ☐

**Exercise** (16.2.50)**.** If $C$ is a smooth curve given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$, show that

$$\int_C \mathbf{r} \cdot d\mathbf{r} = \frac{1}{2} \left[ |\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2 \right].$$

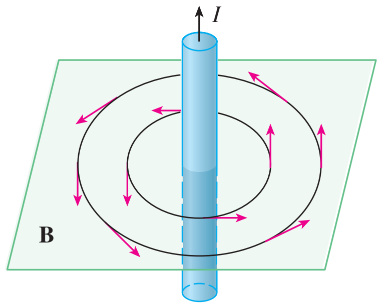
*Solution*: Replace this text with your solution. ☐

**Exercise** (16.2.52)**.** Experiments show that a steady current $I$ in a long wire produces a magnetic field $\mathbf{B}$ that is tangent to any circle that lies in the plane perpendicular to the wire and whose center is the axis of the wire (as in the figure). *Ampère's Law* relates the electric current to its magnetic effects and states that

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where $I$ is the net current that passes through any surface bounded by a closed curve $C$, and $\mu_0$ is a constant called the permeability of free space. By taking $C$ to be a circle with radius $r$, show that the magnitude $B = |\mathbf{B}|$ of the magnetic field at a distance $r$ from the center of the wire is

$$B = \frac{\mu_0 I}{2\pi r}.$$




*Solution*: Replace this text with your solution. □

**Exercise** (16.3.20)**.** Show that the line integral $\int_C \sin y\, dx + (x\cos y - \sin y)\, dy$, where $C$ is any path from $(2,0)$ to $(1,\pi)$, is independent of path and evaluate the integral.

*Solution*: Replace this text with your solution. □

**Exercise** (16.3.29)**.** Show that if the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is conservative and $P$, $Q$, $R$ have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

*Solution*: Replace this text with your solution. □

**Exercise** (16.3.30)**.** Use Exercise 16.3.29 to show that the line integral $\int_C y\, dx + x\, dy + xyz\, dz$ is not independent of path.

*Solution*: Replace this text with your solution. □

**Exercise** (16.3.34)**.** Determine whether or not the set $\{(x,y) \mid (x,y) \neq (2,3)\}$ is (a) open, (b) connected, and (c) simply-connected.

*Solution*: Replace this text with your solution. □

**Exercise** (16.3.35)**.** Let $\mathbf{F}(x,y) = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2+y^2}$.

(a) Show that $\partial P/\partial y = \partial Q/\partial x$.

(b) Show that $\int_C \mathbf{F}\cdot d\mathbf{r}$ is not independent of path. [*Hint:* Compute $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ where $C_1$ and $C_2$ are the upper and lower halves of the circle $x^2+y^2=1$ from $(1,0)$ to $(-1,0)$.] Does this contradict Theorem 16.3.5?

*Solution*: Replace this text with your solution. □

**Exercise** (16.3.34)**.** Determine whether or not the set $\{(x,y) \mid (x,y) \neq (2,3)\}$ is (a) open, (b) connected, and (c) simply-connected.

*Solution*: Replace this text with your solution. □

**Exercise** (16.4.18)**.** A particle starts at the origin, moves along the $x$-axis to $(5, 0)$, then along the quarter-circle $x^2+y^2 = 25$, $x \geq 0$, $y \geq 0$ to the point $(0, 5)$, and then down the $y$-axis back to the origin. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \langle \sin x, \sin y + xy^2 + \frac{1}{3}x^3 \rangle$.

*Solution*: Replace this text with your solution. □

**Exercise** (16.4.19)**.** Use one of the formulas in Theorem 16.4.2 to find the area under one arch of the cycloid $x = t - \sin t, y = 1 - \cos t$.

*Solution*: Replace this text with your solution. □

**Exercise** (16.4.22)**.** Let $D$ be a region bounded by a simple closed path $C$ in the $xy$-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ of $D$ are

$$\bar{x} = \frac{1}{2A} \oint_C x^2 \, dy \qquad \bar{y} = -\frac{1}{2A} \oint_C y^2 \, dx$$

where $A$ is the area of $D$.

*Solution*: Replace this text with your solution. □

**Exercise** (16.4.29)**.** If $\mathbf{F}(x, y) = (-y\mathbf{i}+x\mathbf{j})/(x^2+y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

*Solution*: Replace this text with your solution. □

**Exercise** (16.4.31)**.** Use Green's Theorem to prove the change of variables formula for a double integral (Theorem 15.9.1) for the case where $f(x, y) = 1$:

$$\iint_R dx \, dy = \iint_S \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du \, dv.$$

Here $R$ is the region in the $xy$-plane that corresponds to the region $S$ in the $uv$-plane under the transformation given by $x = g(u, v)$, $y = h(u, v)$.

[*Hint:* Note that the left side is $A(R)$ and apply the first part of Theorem 16.4.2. Convert the line integral over $\partial R$ to a line integral over $\partial S$ and apply Green's Theorem in the $uv$-plane.]

*Solution*: Replace this text with your solution. □

**Exercise** (16.5.33)**.** Use Green's Theorem in the form of Theorem 16.5.5 to prove Green's first identity:

$$\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla f\cdot\nabla g\,dA$$

where $D$ and $C$ satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of $f$ and $g$ exist and are continuous. (The quantity $\nabla g\cdot\mathbf{n} = D_{\mathbf{n}}g$ occurs in the line integral. This is the directional derivative in the direction of the normal vector $\mathbf{n}$ and is called the normal derivative of $g$.)

*Solution*: Replace this text with your solution. □

**Exercise** (16.5.34)**.** Use Green's first identity (Exercise 16.5.33) to prove Green's second identity:

$$\iint_D (f\nabla^2 g - g\nabla^2 f)\,dA = \oint_C (f\nabla g - g\nabla f)\cdot\mathbf{n}\,ds$$

where $D$ and $C$ satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of $f$ and $g$ exist and are continuous.

*Solution*: Replace this text with your solution. □

**Exercise** (16.5.35)**.** Recall from section 14.3 that a function $g$ is called *harmonic* on $D$ if it satisfies Laplace's equation, that is, $\nabla^2 g = 0$ on $D$. Use Green's first identity (with the same hypotheses as in Exercise 16.5.33) to show that if $g$ is harmonic on $D$, then $\oint_C D_{\mathbf{n}}g\,ds = 0$. Here $D_{\mathbf{n}}g$ is the normal derivative of $g$ defined in Exercise 16.5.33.

*Solution*: Replace this text with your solution. □

**Exercise** (16.5.36)**.** Use Green's first identity to show that if $f$ is harmonic on $D$, and if $f(x,y) = 0$ on the boundary curve $C$, then $\iint_D |\nabla f|^2\,dA = 0$. (Assume the same hypotheses as in Exercise 16.5.33.)

*Solution*: Replace this text with your solution. □

**Exercise** (16.5.39)**.** We have seen that all vector fields of the form $\mathbf{F} = \nabla g$ satisfy the equation $\operatorname{curl}\mathbf{F} = \mathbf{0}$ and that all vector fields of the form $\mathbf{F} = \operatorname{curl}\mathbf{G}$ satisfy the equation $\operatorname{div}\mathbf{F} = 0$ (assuming continuity of the appropriate partial derivatives). This suggests the question: are there any equations that all functions of the form $f = \operatorname{div} \mathbf{G}$ must satisfy? Show that the answer to this equation is "No" by proving that *every* continuous function $f$ on $\mathbb{R}^3$ is the divergence of some vector field.

[*Hint:* Let $\mathbf{G}(x, y, z) = \langle g(x, y, z), 0, 0\rangle$, where $g(x, y, z) = \int_0^x f(t, y, z)\, dt$.]

*Solution*: Replace this text with your solution. □

**Exercise** (16.6.34)**.** Find an equation of the tangent place to the parametric surface $x = u^2 + 1$, $y = v^3 + 1$, $z = u + v$ at the point $(5, 2, 3)$.

*Solution*: Replace this text with your solution. □

**Exercise** (16.6.43)**.** Find the area of the surface $z = \frac{2}{3}(x^{3/2}+y^{3/2})$, $0 \leq x \leq 1$, $0 \leq y \leq 1$.

*Solution*: Replace this text with your solution. □

**Exercise** (16.6.47)**.** Find the area of the part of the paraboloid $y = x^2 + z^2$ that lies within the cylinder $x^2 + z^2 = 16$.

*Solution*: Replace this text with your solution. □

**Exercise** (16.6.49)**.** Find the area of the surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \leq u \leq 1$, $0 \leq v \leq 2$.

*Solution*: Replace this text with your solution. □

**Exercise** (16.6.53)**.** Find the area of the part of the surface $z = \ln(x^2 + y^2 + 2)$ that lies above the disk $x^2 + y^2 \leq 1$ correct to four decimal places by expressing the area in terms of a single integral and using your calculator to estimate the integral.

*Solution*: Replace this text with your solution. □

**Exercise** (16.7.17)**.** Evaluate the surface integral $\iint_S (x^2z + y^2z)\, dS$ where $S$ is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$.

*Solution*: Replace this text with your solution. ☐

**Exercise** (16.7.26)**.** Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the vector field $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}$ where $S$ is the is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, oriented downward. In other words, find the flux of $\mathbf{F}$ across $S$.

*Solution*: Replace this text with your solution. ☐

**Exercise** (16.7.37)**.** Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to the one in Remark 3 for the case where $S$ is given by $y = h(x, z)$ and $\mathbf{n}$ is the unit normal that points toward the left.

*Solution*: Replace this text with your solution. ☐

**Exercise** (16.7.40)**.** Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$, $1 \leq z \leq 4$, if its density function is $\rho(x, y, z) = 10 - z$.

*Solution*: Replace this text with your solution. ☐

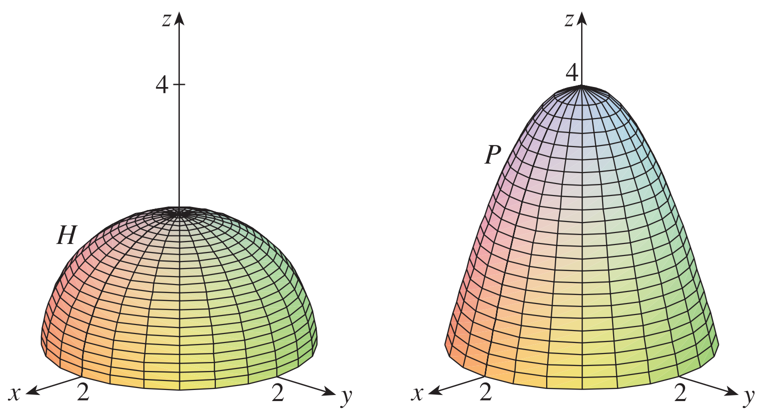
**Exercise** (16.7.46)**.** Use Gauss's Law to find the charge enclosed by the cube with vertices $(\pm 1, \pm 1, \pm 1)$ if the electric field is

$$\mathbf{E}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

*Solution*: Replace this text with your solution. ☐

**Exercise** (16.8.1)**.** A hemisphere $H$ and a portion $P$ of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$




*Solution*: Replace this text with your solution. □

**Exercise** (16.8.7)**.** Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = (x + y^2)\mathbf{i} + (y + z^2)\mathbf{j} + (z + x^2)\mathbf{k}$ and $C$ is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ oriented counterclockwise as viewed from above.

*Solution*: Replace this text with your solution. □

**Exercise** (16.8.13)**.** Verify that Stokes' Theorem is true for the vector field $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} - 2\mathbf{k}$, where $S$ is the cone $z^2 = x^2 + y^2$, $0 \leq z \leq 4$, oriented downward.

*Solution*: Replace this text with your solution. □

**Exercise** (16.8.16)**.** Let $C$ be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z\, dx - 2x\, dy + 3y\, dz$$

depends only on the area of the region enclosed by $C$ and not on the shape of $C$ or its location in the plane.

*Solution*: Replace this text with your solution. □

**Exercise** (16.8.19)**.** If $S$ is a sphere and $\mathbf{F}$ satisfies the hypotheses of Stokes' Theorem, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

*Solution*: Replace this text with your solution. □

**Exercise** (16.9.18)**.** Let $\mathbf{F}(x, y, z) = z \tan^{-1}(y^2)\mathbf{i} + z^3 \ln(x^2 + 1)\mathbf{j} + z\mathbf{k}$. Find the flux of $\mathbf{F}$ across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward.

*Solution*: Replace this text with your solution. □

**Exercise** (16.9.24)**.** Use the Divergence Theorem to evaluate

$$\iint_S (2x + 2y + z^2)\, dS$$

where $S$ is the sphere $x^2 + y^2 + z^2 = 1$.

*Solution*: Replace this text with your solution. □

**Exercise** (16.9.25)**.** Prove the identity

$$\iint_S \mathbf{a} \cdot \mathbf{n}\, dS = 0,$$

where $\mathbf{a}$ is a constant vector, assuming that $S$ satisfies the conditions of the Divergence Theorem and the components of the vector field have continuous second-order partial derivatives.

*Solution*: Replace this text with your solution. □

**Exercise** (16.9.31)**.** Suppose $S$ and $E$ satisfy the conditions of the Divergence Theorem and $f$ is a scalar function with continuous partial derivatives. Prove that

$$\iint_S f\mathbf{n}\, dS = \iiint_E \nabla f\, dV.$$

These surfaces and triple integrals of vector functions are vectors defined by integrating each component function.
[*Hint:* Start by applying the Divergence Theorem to $\mathbf{F} = f\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector.]

*Solution*: Replace this text with your solution. □

**Exercise** (16.9.32)**.** A solid occupies a region $E$ with surface $S$ and is immersed in a liquid with constant density $\rho$. We set up a coordinate system so that the $xy$-plane coincides with the surface of the liquid, and positive values of $z$ are measured downward into the liquid. Then the pressure at depth $z$

is $p = \rho g z$, where $g$ is the acceleration due to gravity (see Section 8.3). The total buoyant force on the solid due to the pressure distribution is given by the surface integral

$$\mathbf{F} = -\iint_S \rho \mathbf{n} \, dS$$

where $\mathbf{n}$ is the outer unit normal. Use the result of Exercise 16.9.31 to show that $\mathbf{F} = -W\mathbf{k}$, where $W$ is the weight of the liquid displaced by the solid. (Note that $\mathbf{F}$ is directed upward because $z$ is directed downward.) The result is Archimedes' Principle: The buoyant force on an object equals the weight of the displaced liquid.

*Solution*: Replace this text with your solution. □