Calculus I Notes

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Contents

Chapter 2

Limits and Derivatives

2.1 The Tangent and Velocity Problems

Remark 1. A tangent to a curve is a line that that touches the curve. A secant is a line that cuts a curve more than once.

Example 1. Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Example 2. A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heartbeats after t minutes. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute. The monitor estimates this value by calculating the slope of a secant line. Use the data to draw the graph of this function and estimate the patient's heart rate after 42 minutes.

t (min)	Heartbeats
36	2530
38	2661
40	2806
42	2948
44	3080

Example 3. Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds. [If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law that the distance fallen by any freely falling body is proportional to the square of the time it has been falling is expressed by the equation $s(t) = 4.9t^2$.]

2.2 The Limit of a Function

Definition 2.2.1. Suppose $f(x)$ is defined when x is near the number a. Then we write

$$\lim_{x\to a} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a but not equal to a.

Example 1. Guess the value of $\displaystyle\lim_{x\to 3} \frac{x^2-3x}{x^2-9}$.

Example 2. Estimate the value of $\displaystyle\lim_{t\to 4} \frac{\ln t - \ln 4}{t-4}$.

Example 3. Guess the value of $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$.

Example 4. Investigate $\lim_{x\to 0} \sin\dfrac{\pi}{x}$.

Example 5. Find $\lim_{x\to 0}\left(x^3 + \dfrac{\cos 5x}{10,000}\right)$.

Definition 2.2.2. We write

$$\lim_{x\to a^-} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x less than a. Similarly, if we require that x be greater than a, we write

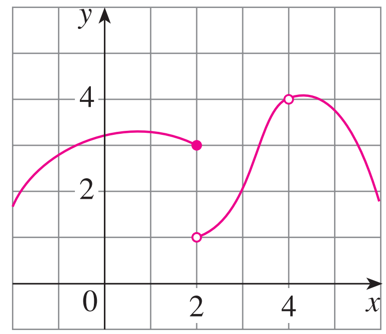
$$\lim_{x\to a^+} f(x) = L.$$

Example 6. Investigate the limit as t approaches 0 of the Heaviside function H, defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

Remark 1. $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

Example 7. Use the graph of f to state the values (if they exist) of the following:



(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$

(c) $\lim_{x\to 2} f(x)$ (d) $f(2)$

(e) $\lim_{x\to 4} f(x)$ (f) $f(4)$

Definition 2.2.3. Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a. Similarly,

$$\lim_{x\to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

Example 8. Find $\lim_{x\to 1} \dfrac{2-x}{(x-1)^2}$ if it exists.

Definition 2.2.4. The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

Example 9. Find $\displaystyle\lim_{x\to 1^+} \frac{1}{x^3-1}$ and $\displaystyle\lim_{x\to 1^-} \frac{1}{x^3-1}$.

Example 10. Find the vertical asymptotes of $f(x) = \tan x$.

2.3 Calculating Limits Using the Limit Laws

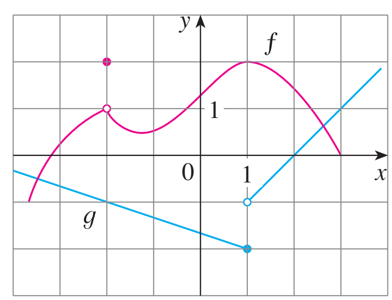
Theorem 2.3.1 (Limit Laws)**.** *Suppose that c is a constant and the limits*

$$\lim_{x\to a} f(x) \qquad \textit{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$
5. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0$

Example 1. Use the Limit Laws and the graphs of f and g to evaluate the following limits, if they exist.



(a) $\lim_{x\to -2}[f(x)+5g(x)]$

(b) $\lim_{x\to 1}[f(x)g(x)]$

(c) $\lim_{x\to 2}\dfrac{f(x)}{g(x)}$

Theorem 2.3.2 (Power and Root Laws)**.** *By repeatedly applying the Product Law and using some basic intuition we obtain the following:*

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ *where n is a positive integer*
7. $\lim_{x \to a} c = c$
8. $\lim_{x \to a} x = a$
9. $\lim_{x \to a} x^n = a^n$ *where n is a positive integer*
10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ *where n is a positive integer*
    (If n is even, we assume that $a > 0$.)
11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ *where n is a positive integer*
    $\left[\textit{If } n \textit{ is even, we assume that } \lim_{x \to a} f(x) > 0.\right]$

Example 2. Evaluate the following limits and justify each step.

(a) $\lim_{x \to -3} (2x^3 + 6x^2 - 9)$

(b) $\lim_{t \to 7} \dfrac{3t^2 + 1}{t^2 - 5t + 2}$

Theorem 2.3.3 (Direct Substitution Property)**.** *If f is a polynomial or a rational function and a is in the domain of f, then*

$$\lim_{x \to a} f(x) = f(a).$$

Example 3. Find $\displaystyle\lim_{t \to 4} \frac{t^2 - 2t - 8}{t - 4}$.

Remark 1. If $f(x) = g(x)$ when $x \neq a$, then $\displaystyle\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

Example 4. Find $\displaystyle\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} x + 1 & \text{if } x \neq 1, \\ \pi & \text{if } x = 1. \end{cases}$$

Example 5. Evaluate $\displaystyle\lim_{h \to 0} \frac{(h - 3)^2 - 9}{h}$.

Example 6. Find $\lim_{h\to 0} \dfrac{\sqrt{9+h}-3}{h}$.

Example 7. Show that $\lim_{x\to 0} |x| = 0$.

Example 8. Prove that $\lim_{x\to 0} \dfrac{|x|}{x}$ does not exist.

Example 9. If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4, \\ 8-2x & \text{if } x < 4. \end{cases}$$

determine whether $\lim_{x\to 4} f(x)$ exists.

Example 10. The greatest integer function is defined by $[\![x]\!]$ = the largest integer that is less than or equal to x. (For instance, $[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}]\!] = 1$, $[\![-\frac{1}{2}]\!] = -1$.) Show that $\lim_{x\to 3}[\![x]\!]$ does not exist.

Theorem 2.3.4. *If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then*

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

Theorem 2.3.5 (The Squeeze Theorem)**.** *If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L.$$

Example 11. Show that $\lim_{x\to 0} x^4 \cos\dfrac{2}{x} = 0$.

2.4 The Precise Definition of a Limit

Definition 2.4.1. Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Example 1. Use a graph to find a number δ such that if x is within δ of 1, then $f(x) = x^3 - 5x + 6$ is within 0.2 of 2.

Example 2. Prove that $\lim_{x \to 3}(4x - 5) = 7$.

Definition 2.4.2.

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Similarly,

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Example 3. Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Example 4. Prove that $\lim_{x \to 3} x^2 = 9$.

Definition 2.4.3. Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) > M.$$

Similarly,

$$\lim_{x \to a} f(x) = -\infty$$

means that for every negative number N there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) < N.$$

Example 5. Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

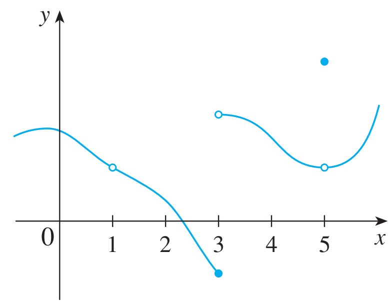
2.5 Continuity

Definition 2.5.1. A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a).$$

We say that f is discontinuous at a (or f has a discontinuity at a) if f is not continuous at a.

Example 1. Use the graph of the function f to determine the numbers at which f is discontinuous.



Example 2. Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

Definition 2.5.2. A function f is <u>continuous from the right at a number a</u> if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is <u>continuous from the left at a</u> if

$$\lim_{x \to a^-} f(x) = f(a).$$

Example 3. In which direction(s) is the function $f(x) = [\![x]\!]$ continuous?

Definition 2.5.3. A function f is <u>continuous on an interval</u> if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.)

Example 4. Show that the function $f(x) = x + \sqrt{x-4}$ is continuous on the interval $[4, \infty)$.

Theorem 2.5.1. *If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:*

1. $f + g$ *2.* $f - g$ *3.* cf

4. fg *5.* $\dfrac{f}{g}$ *if* $g(a) \neq 0$

Proof. All of these results follow from the Limit Laws. For example, $f+g$ is continuous at a because

$$\begin{aligned}\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\ &= f(a) + g(a) \\ &= (f+g)(a).\end{aligned}$$

□

Theorem 2.5.2. *(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.*

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Proof. (a) Let

$$P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

be a polynomial where $c_0, c_1, \ldots, c_n$ are constants. Then

$$\lim_{x\to a} x^m = a^m \qquad m = 1, 2, \ldots, n$$

implies that the function $f(x) = x^m$ is continuous. Since

$$\lim_{x\to a} c_0 = c_0,$$

the constant function is continuous as well, and therefore the product function $g(x) = cx^m$ is continuous. Since P is a sum of functions of this form, it is continuous as well.

(b) Rational functions are quotients of polynomials, i.e.,

$$f(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials. Thus the above result implies that they are continuous on their domains. $\square$

Example 5. Find $\lim\limits_{x\to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$.

Theorem 2.5.3. *The following types of functions are continuous at every number in their domains:*

- *polynomials*
- *rational functions*
- *root functions*
- *trigonometric functions*
- *inverse trigonometric functions*
- *exponential functions*
- *logarithmic functions*

Example 6. Where is the function $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?

Example 7. Evaluate $\lim\limits_{x\to \pi} \dfrac{\sin x}{2 + \cos x}$.

Theorem 2.5.4. *If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$, i.e.,*

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

Proof. Let $\varepsilon > 0$. Since f is continuous at b, we have $\lim_{y\to b} f(y) = f(b)$ and so there exists $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |y-b| < \delta_1 \qquad \text{then} \qquad |f(y) - f(b)| < \varepsilon.$$

Since $\lim_{x\to a} g(x) = b$, there exists $\delta > 0$ such that

$$\text{if} \quad 0 < |x-a| < \delta \qquad \text{then} \qquad |g(x) - b| < \delta_1.$$

By letting $y = g(x)$ in the first statement, we get that $0 < |x-a| < \delta$ implies that $\big|f(g(x)) - f(b)\big| < \varepsilon$, i.e., $\lim_{x\to a} f(g(x)) = f(b)$. $\square$

Example 8. Evaluate $\lim_{x\to 1} \ln\left(\dfrac{5-x^2}{1+x}\right)$.

Theorem 2.5.5. *If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f\circ g)(x) = f(g(x))$ is continuous at a.*

Proof. Since g is continuous at a, we have

$$\lim_{x\to a} g(x) = g(a).$$

Since f is continuous at $g(a)$, we have

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(g(a)),$$

which means $f \circ g$ is continuous. $\square$

Example 9. Where are the following functions continuous?

(a) $f(x) = \dfrac{1}{\sqrt{1 - \sin x}}$

(b) $y = \arctan \dfrac{1}{x}$

Theorem 2.5.6 (Intermediate Value Theorem)**.** *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.*

Example 10. Show that there is a root of the equation $-x^3 + 4x + 1 = 0$ between -1 and 0.

2.6 Limits at Infinity

Definition 2.6.1. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Definition 2.6.2. Let f be a function defined on some interval $(-\infty, a)$. Then

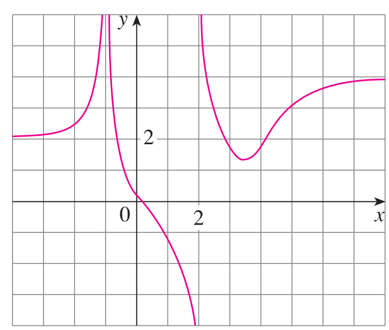
$$\lim_{x\to-\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

Definition 2.6.3. The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

Example 1. Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown.



Example 2. Find $\displaystyle\lim_{x\to\infty} \frac{1}{x}$ and $\displaystyle\lim_{x\to-\infty} \frac{1}{x}$.

Theorem 2.6.1. *If $r > 0$ is a rational number, then*

$$\lim_{x \to \infty} \frac{1}{x^r} = 0.$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0.$$

Proof. By extending the limit laws to limits at infinity we get

$$\begin{aligned}
\lim_{x \to \infty} \frac{1}{x^r} &= \lim_{x \to \infty} \left[\frac{1}{x}\right]^r = \left[\lim_{x \to \infty} \frac{1}{x}\right]^r = 0^r = 0 \\
\lim_{x \to -\infty} \frac{1}{x^r} &= \lim_{x \to -\infty} \left[\frac{1}{x}\right]^r = \left[\lim_{x \to -\infty} \frac{1}{x}\right]^r = 0^r = 0.
\end{aligned}$$

□

Example 3. Evaluate

$$\lim_{x \to \infty} \frac{3x^3 - 8x + 2}{4x^3 - 5x^2 - 2}.$$

Example 4. Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}.$$

Example 5. Compute $\lim_{t\to\infty}(\sqrt{25t^2+2}-5t)$.

Example 6. Evaluate $\lim_{x\to 0^+} \tan^{-1}(\ln x)$.

Example 7. Evaluate $\lim_{x\to(\pi/2)^+} e^{\sec x}$.

Example 8. Evaluate $\lim_{x\to\infty} \cos x$.

Example 9. Find $\lim_{x\to\infty} x^5$ and $\lim_{x\to-\infty} x^5$.

Example 10. Find $\lim_{x\to\infty} (x - \sqrt{x})$.

Example 11. Find $\lim_{x\to\infty} \dfrac{x^2+x}{3-x}$.

Example 12. Sketch the graph of $y = (3-x)(1+x)^2(1-x)^4$ by finding its intercepts and its limits as $x \to \infty$ and as $x \to -\infty$.

Definition 2.6.4. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Definition 2.6.5. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x < N \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Example 13. Use a graph to find a number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1.$$

Example 14. Prove that $\displaystyle\lim_{x\to\infty} \frac{1}{x} = 0$.

Definition 2.6.6. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad f(x) > M.$$

Definition 2.6.7. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = -\infty$$

means that for every negative number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad f(x) < M.$$

Definition 2.6.8. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = \infty$$

means that for every positive number M there is a corresponding negative number N such that

$$\text{if} \quad x < N \qquad \text{then} \qquad f(x) > M.$$

Definition 2.6.9. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = -\infty$$

means that for every negative number M there is a corresponding negative number N such that

$$\text{if} \quad x < N \qquad \text{then} \qquad f(x) < M.$$

2.7 Derivatives and Rates of Change

Definition 2.7.1. The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Example 1. Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Example 2. Use the alternative expression for the slope of a tangent line

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

to find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

Definition 2.7.2. A function f describing the motion of an object along a straight line is called a position function and has velocity

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

at time $t = a$.

Example 3. Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground. Recall that the distance (in meters) fallen after t seconds is $4.9t^2$.
(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Definition 2.7.3. The derivative of a function f at a number a, denoted by $f'(a)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

or equivalently

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

Example 4. Find the derivative of the function $f(x) = 2x^2 - 5x + 3$ at the numbers (a) 2 and (b) a.

Example 5. Find the derivative of the function $f(x) = 1/\sqrt{x}$ at the number a ($a > 0$).

Example 6. Find an equation of the tangent line to the parabola $y = 2x^2 - 5x + 3$ at the point $(3, 6)$.

Definition 2.7.4. Suppose y is a quantity that depends on another quantity x. Then y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the <u>increment</u> of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1).$$

The <u>average rate of change of y with respect x</u> over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

and the <u>instantaneous rate of change of y with respect to x</u> is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x).$$

Example 7. The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars.
(a) What is the meaning of the derivative $f'(x)$? What are its units?

(b) What does the statement $f'(800) = 17$ mean?

(c) Do you think the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

Example 8. Let $D(t)$ be the US national debt at time t. The table gives approximate values of this function by providing end of year estimates, in billions of dollars, from 1985 to 2010. Interpret and estimate the value of $D'(2000)$.

t	$D(t)$
1985	1945.9
1990	3364.8
1995	4988.7
2000	5662.2
2005	8170.4
2010	14,025.2

Source: US Dept. of the Treasury

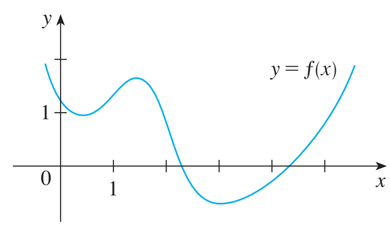
2.8 The Derivative as a Function

Definition 2.8.1. The derivative of a function f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if this limit exists.

Example 1. The graph of a function f is given. Use it to sketch the graph of the derivative f'.



Example 2. (a) If $A(p) = 4p^3 + 3p$, find a formula for $A'(p)$.

(b) Illustrate this formula by comparing the graphs of A and A'.

Example 3. If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

Example 4. Find g' if $g(u) = \dfrac{u+1}{4u-1}$.

Definition 2.8.2. The symbols D and d/dx are called <u>differentiation operators</u> and are used as follows:

$$f'(x) = y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x).$$

For fixed a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_{x=a}$$

Definition 2.8.3. A function f is <u>differentiable at a</u> if $f'(a)$ exists. It is <u>differentiable on an open interval</u> (a,b) [or (a,∞) or $(-\infty,a)$ or $(-\infty,\infty)$] if it is differentiable at every number in the interval.

Example 5. Where is the function $f(x) = |x|$ differentiable?

Theorem 2.8.1. *If f is differentiable at a, then f is continuous at a.*

Proof. If f is differentiable at a, we have

$$\begin{aligned}\lim_{x\to a}[f(x)-f(a)] &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}(x-a)\\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a)\\ &= f'(a)\cdot 0 = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}[f(a)+(f(x)-f(a))]\\ &= \lim_{x\to a} f(a)+\lim_{x\to a}[f(x)-f(a)]\\ &= f(a)+0 = f(a).\end{aligned}$$

□

Definition 2.8.4. If the derivative f' of a function f has a derivative of its own we call it the <u>second derivative</u> of f and denote it by

$$(f')' = f'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Example 6. If $A(p) = 4p^3 + 3p$, find and interpret $A''(p)$.

Definition 2.8.5. The instantaneous rate of change of velocity with respect to time is called the <u>acceleration</u> $a(t)$ of an object. It is the derivative of the velocity function, and therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t).$$

Definition 2.8.6. The third derivative f''' is the derivative of the second derivative, denoted by

$$(f'')' = f'''.$$

Definition 2.8.7. The instantaneous rate of change of acceleration with respect to time is called the jerk $j(t)$ of an object. It is the derivative of the acceleration function, and therefore the third derivative of the position function:

$$j(t) = a'(t) = v''(t) = s'''(t).$$

Definition 2.8.8. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Example 7. If $A(p) = 4p^3 + 3p$, find $A'''(p)$ and $A^{(4)}(p)$.

Chapter 3

Differentiation Rules

3.1 Derivatives of Polynomials and Exponentials

Theorem 3.1.1. *The derivative of a constant function* $f(x) = c$ *is* 0*, i.e.,*

$$\frac{d}{dx}(c) = 0.$$

Proof.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$ □

Theorem 3.1.2.

$$\frac{d}{dx}(x) = 1 \qquad \frac{d}{dx}(x^2) = 2x \qquad \frac{d}{dx}(x^3) = 3x^2 \qquad \frac{d}{dx}(x^4) = 4x^3$$

Proof. All of these follow directly from the definition of the derivative, as above. □

Theorem 3.1.3 (The Power Rule)**.** *If* n *is a positive integer, then*

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof. Since

$$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}),$$

we have

$$\begin{aligned}
f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{x^n-a^n}{x-a} \\
&= \lim_{x\to a}(x^{n-1}+x^{n-2}a+\cdots+xa^{n-2}+a^{n-1}) \\
&= a^{n-1}+a^{n-2}a+\cdots+aa^{n-2}+a^{n-1} \\
&= \underbrace{a^{n-1}+a^{n-1}+\cdots+a^{n-1}+a^{n-1}}_{n} \\
&= na^{n-1}.
\end{aligned}$$

□

Example 1. Find the derivative of each of the following:

(a) $f(x) = x^5$

(b) $y = x^{555}$

(c) $y = t^7$

(d) $f(r) = r^2$

Theorem 3.1.4 (The Power Rule (General Version))**.** *If n is any real number, then*

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Example 2. Differentiate:

(a) $f(x) = \dfrac{1}{x^3}$

(b) $y = \sqrt[3]{x^4}$

Definition 3.1.1. The <u>normal line</u> to a curve C at a point P is the line through P that is perpendicular to the tangent line at P.

Example 3. Find equations of the tangent line and normal line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$.

Theorem 3.1.5 (The Constant Multiple Rule)**.** *If c is a constant and f is a differentiable function, then*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x).$$

Proof. Let $g(x) = cf(x)$. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} \\ &= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right] \\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= cf'(x). \end{aligned}$$ □

Example 4. Find:

(a) $\dfrac{d}{dx}(10x^3)$

(b) $\dfrac{d}{dx}(-x)$

Theorem 3.1.6 (The Sum Rule)**.** *If f and g are both differentiable, then*

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Proof. Let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right] \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) + g'(x).
\end{aligned}$$

□

Theorem 3.1.7 (The Difference Rule)**.** *If f and g are both differentiable, then*

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x).$$

Example 5. Find $\dfrac{d}{dx}(x^9 - 8x^7 - 2x^4 + 7x^3 + 2x + 6)$.

Example 6. Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

Example 7. The equation of motion of a particle is $s = t^4 - 2t^3 + t^2 - t$, where s is measured in meters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

Definition 3.1.2. e is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

Theorem 3.1.8. $\frac{d}{dx}(e^x) = e^x$.

Example 8. If $f(r) = e^r + r^e$, find f' and f''.

Example 9. At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $3x - y = 5$?

3.2 The Product and Quotient Rules

Theorem 3.2.1 (The Product Rule)**.** *If f and g are both differentiable, then*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

Proof. By the definition of the derivative on the product,

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)[g(x+h) - g(x)]}{h} + \lim_{h\to 0}\frac{g(x)[f(x+h) - f(x)]}{h} \\
&= \lim_{h\to 0} f(x+h) \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\
&= f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].
\end{aligned}$$

□

Example 1. (a) If $f(x) = xe^x$, find $f'(x)$.

(b) Find the nth derivative, $f^{(n)}(x)$.

Example 2. Differentiate the function $J(u) = \left(\frac{1}{u} + \frac{1}{u^2}\right)\left(u + \frac{1}{u}\right)$.

Example 3. If $f(x) = e^x g(x)$, where $g(0) = 2$ and $g'(0) = 5$, find $f'(0)$.

Theorem 3.2.2 (The Quotient Rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Proof. Similar to the Product Rule, except we add and subtract $f(x)g(x)$ in the numerator when applying the definition of the derivative. □

Example 4. Let $f(x) = \dfrac{ax+b}{cx+d}$. Find f'.

Example 5. Find an equation of the tangent line to the curve $y = (1+x)/(1+e^x)$ at the point $(0, \frac{1}{2})$.

3.3 Derivatives of Trigonometric Functions

Theorem 3.3.1. *The derivative of the sine function is the cosine function, i.e.,*

$$\frac{d}{dx}(\sin x) = \cos x.$$

Example 1. Differentiate $y = x^3 \sin x$.

Theorem 3.3.2. *The derivative of the cosine function is the negative sine function, i.e.,*

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Theorem 3.3.3. *The derivative of the tangent function is the square of the secant function, i.e.,*

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

Proof. By the Quotient Rule,

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}\\ &= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x} = \sec^2 x.\end{aligned}$$

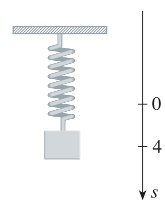
□

Theorem 3.3.4. *The derivatives of the trigonometric functions are*

$$
\begin{aligned}
\frac{d}{dx}(\sin x) &= \cos x & \qquad \frac{d}{dx}(\csc x) &= -\csc x \cot x \\
\frac{d}{dx}(\cos x) &= -\sin x & \qquad \frac{d}{dx}(\sec x) &= \sec x \tan x \\
\frac{d}{dx}(\tan x) &= \sec^2 x & \qquad \frac{d}{dx}(\cot x) &= -\csc^2 x.
\end{aligned}
$$

Example 2. Differentiate $f(x) = \dfrac{\sec x}{1+\tan x}$. For what values of x does the graph of f have a horizontal tangent?

Example 3. An object at the end of a vertical spring is stretched to 4 cm beyond its reset position and released at time $t = 0$. (See the figure and note that the downward direction is positive.) Its position at time t is



$$s = f(t) = 4\cos t.$$

Find the velocity and acceleration at time t and use them to analyze the motion of the object.

Example 4. Find the 99th derivative of $\sin x$.

Example 5. Find $\lim_{x \to 0} \dfrac{\sin 5x}{3x}$.

Example 6. Find $\lim_{x \to 0} \dfrac{\sin x}{\sin \pi x}$.

Example 7. Find $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta + \tan \theta}$.

3.4 The Chain Rule

Theorem 3.4.1 (The Chain Rule)**.** *If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product*

$$F'(x) = f'(g(x)) \cdot g'(x).$$

Example 1. Find $F'(x)$ if $F(x) = \sqrt{x^3 + 2}$.

Example 2. Differentiate (a) $y = \cos(x^2)$ and (b) $y = \cos^2 x$.

Theorem 3.4.2 (The Power Rule Combined with the Chain Rule)**.** *If n is any real number and $u = g(x)$ is differentiable, then*

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

Example 3. Differentiate $y = (x^5 + 3x^2 - x)^{50}$.

Example 4. Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{e^x + 1}}$.

Example 5. Find the derivative of the function

$$g(u) = \left(\frac{u^3 - 1}{u^3 + 1}\right)^8.$$

Example 6. Differentiate $F(x) = (4x+5)^3(x^2-2x+5)^4$.

Example 7. Differentiate $y = e^{\tan\theta}$.

Theorem 3.4.3. *The derivative of the exponential function is*

$$\frac{d}{dx}(b^x) = b^x \ln b.$$

Proof. Since

$$b^x = (e^{\ln b})^x = e^{(\ln b)x},$$

the Chain Rule gives

$$\begin{aligned}
\frac{d}{dx}(b^x) &= \frac{d}{dx}(e^{(\ln b)x}) \\
&= e^{(\ln b)x}\frac{d}{dx}(\ln b)x \\
&= e^{(\ln b)x} \cdot \ln b \\
&= b^x \ln b.
\end{aligned}$$

□

Example 8. Find the derivative of (a) $g(x) = 3^x$ and (b) $h(x) = 5^{\sqrt{x}}$.

Example 9. Find $f'(t)$ if $f(t) = \tan(\sec(\cos t))$.

Example 10. Differentiate $y = e^{\sin^2(x^2)}$.

3.5 Implicit Differentiation

Definition 3.5.1. Implicit differentiation is the method of differentiation both sides of an equation with respect to x, and then solving the equation for y' when $y = f(x)$.

Example 1. (a) If $x^2 + y^2 = 169$, find $\frac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 169$ at the point $(5, 12)$.

Example 2. (a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

(c) At what point in the first quadrant is the tangent line horizontal?

Example 3. Find y' if $\tan(x - y) = 2xy^3 + 1$.

Example 4. Find y'' if $x^3 - y^3 = 7$.

3.6 Derivatives of Logarithmic and Inverse Trigonometric Functions

Theorem 3.6.1. *The derivative of the logarithm function is*

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}.$$

Proof. Let $y = \log_b x$. Then $b^y = x$, so by differentiating we get

$$\begin{aligned} b^y &= x \\ b^y (\ln b) \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{b^y \ln b} \\ &= \frac{1}{x \ln b}. \end{aligned}$$

□

Theorem 3.6.2. *The derivative of the natural logarithm is*

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Example 1. Differentiate $y = \log_8(x^2 + 3x)$.

Example 2. Find $\dfrac{d}{dx} \ln(\cos x)$.

Example 3. Differentiate $g(t) = \sqrt{1 + \ln t}$.

Example 4. Differentiate $y = \log_{10} \sec x$.

Example 5. Find $\dfrac{d}{dx} \ln \dfrac{x^a}{b^x}$.

Example 6. Find $f'(x)$ if $f(x) = \ln |x|$.

Definition 3.6.1. Logarithmic differentiation is the method of calculating derivatives of functions by taking logarithms, differentiating implicitly, and then solving the resulting equation for the derivative.

Example 7. Differentiate $y = \dfrac{e^{-x}\cos^2 x}{x^2+x+1}$.

Theorem 3.6.3 (The Power Rule). *If n is any real number and $f(x) = x^n$, then*

$$f'(x) = nx^{n-1}.$$

Proof. Let $y = x^n$. By logarithmic differentiation we get

$$\begin{aligned} y &= x^n \\ \ln|y| &= \ln|x|^n \\ &= n\ln|x| \qquad x \neq 0 \\ \frac{y'}{y} &= \frac{n}{x} \\ y' &= n\frac{y}{x} \\ &= n\frac{x^n}{x} \\ &= nx^{n-1}. \end{aligned}$$

□

Example 8. Differentiate $y = (\sqrt{x})^x$.

Theorem 3.6.4. *The number e can be defined as the limit*

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Proof. If $f(x) = \ln x$, then $f'(1) = 1$, so

$$\begin{aligned} f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{x\to 0}\frac{f(1+x)-f(1)}{x} \\ &= \lim_{x\to 0}\frac{\ln(1+x)-\ln 1}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x) \\ &= \lim_{x\to 0}\ln(1+x)^{1/x} = 1. \end{aligned}$$

Thus

$$e = e^1 = e^{\left(\lim_{x\to 0}\ln(1+x)^{1/x}\right)} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}} = \lim_{x\to 0}(1+x)^{1/x}.$$

Then if we let $n = 1/x$, $n\to\infty$ as $x\to 0^+$, so we are done. $\square$

Theorem 3.6.5. *The derivative of the arcsine function is*

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

Proof. Since $y = \sin^{-1} x$ means $\sin y = x$ and $-\pi/2 \leq y \leq \pi/2$, we have $\cos y \geq 0$. Thus we can differentiate to obtain

$$\begin{aligned} \sin y &= x \\ \cos y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y} \\ &= \frac{1}{\sqrt{1-\sin^2 y}} \\ &= \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

□

Theorem 3.6.6. *The derivative of the arctangent function is*

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

Proof. If $y = \tan^{-1} x$, then $\tan y = x$. Differentiating then gives us

$$\begin{aligned} \tan y &= x \\ \sec^2 y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} \\ &= \frac{1}{1+\tan^2 y} \\ &= \frac{1}{1+x^2}. \end{aligned}$$

□

Example 9. Differentiate

(a) $y = \dfrac{1}{\tan^{-1} x}$

(b) $h(x) = (\arcsin x) \ln x.$

Theorem 3.6.7. *The derivatives of the Inverse Trigonometric Functions are*

$$\begin{aligned} \frac{d}{dx}(\sin^{-1} x) &= \frac{1}{\sqrt{1-x^2}} & \qquad \frac{d}{dx}(\csc^{-1} x) &= -\frac{1}{x\sqrt{x^2-1}} \\ \frac{d}{dx}(\cos^{-1} x) &= -\frac{1}{\sqrt{1-x^2}} & \qquad \frac{d}{dx}(\sec^{-1} x) &= \frac{1}{x\sqrt{x^2-1}} \\ \frac{d}{dx}(\tan^{-1} x) &= \frac{1}{1+x^2} & \qquad \frac{d}{dx}(\cot^{-1} x) &= -\frac{1}{1+x^2}. \end{aligned}$$

Theorem 3.6.8. *Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Then f^{-1} has derivative*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.

Proof. Since $(f \circ f^{-1})(x) = x$, we have, by the chain rule,

$$\begin{aligned}(f \circ f^{-1})(x) &= x \\ (f \circ f^{-1})'(x) &= 1 \\ f'(f^{-1}(x))(f^{-1})'(x) &= 1 \\ (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))}.\end{aligned}$$

□

Example 10. If $f(4) = 5$ and $f'(4) = \frac{2}{3}$, find $(f^{-1})'(5)$.

3.7 Rates of Change in the Sciences

Example 1. The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters.

(a) Find the velocity at time t.

(b) What is the velocity after 2 s? After 4 s?

(c) When is the particle at rest?

(d) When is the particle moving forward (that is, in the positive direction)?

(e) Draw a diagram to represent the motion of the particle.

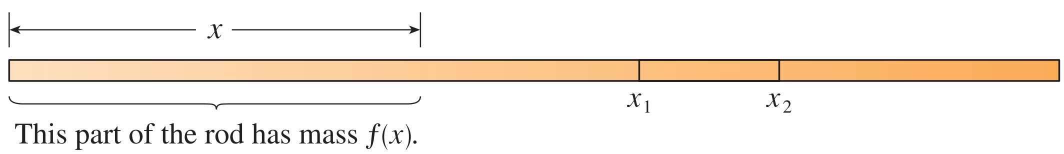
(f) Find the total distance traveled by the particle during the first five seconds.

(g) Find the acceleration at time t and after 4 s.

(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$.

(i) When is the particle speeding up? When is it slowing down?

Example 2. If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length ($\rho = m/l$) and measured in kilograms per meter. Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point x is $m = f(x)$, as shown in the figure.



In this case the average density is the average rate of change over a given interval, and the linear density is the limit of these average densities.
If $m = f(x) = \sqrt{x}$, where x is measured in meters and m in kilograms, find the average density of the part of the rod given by $1 \le x \le 1.2$ and the density at $x = 1$.

Example 3. The average current during a time interval is the average rate of change of the net charge over that interval, and the current at a given time is the limit of the average current (the rate at which charge flows through a surface, measured in units of charge per unit time). The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds) is given by $Q(t) = t^3 - 2t^2 + 6t + 2$. [The unit of current is an ampere (1 A = 1 C/s).] Find the current when

(a) $t = 0.5$ s

(b) $t = 1$ s.

At what time is the current lowest?

Example 4. The concentration of a reactant A is the number of moles (1 mole $= 6.022 \times 10^{23}$ molecules) per liter and is denoted by [A] for a chemical reaction

$$\mathrm{A} + \mathrm{B} \rightarrow \mathrm{C}.$$

The average rate of reaction during a time interval is the average rate of change of the concentration of the product [C] over that interval, and the rate of reaction at a given time is the limit of the average rate of reaction.
If one molecule of a product C is formed from one molecule of a reactant A and one molecule of a reactant B, and the initial concentrations of A and B have a common value $[\mathrm{A}] = [\mathrm{B}] = a$ moles/L, then

$$[\mathrm{C}] = \frac{a^2kt}{akt+1}$$

where k is a constant.

(a) Find the rate of reaction at time t.

(b) Show that if $x = [\mathrm{C}]$, then

$$\frac{dx}{dt} = k(a-x)^2.$$

(c) What happens to the concentration as $t \to \infty$?

(d) What happens to the rate of reaction as $t \to \infty$?

(e) What do the results of parts (c) and (d) mean in practical terms?

Example 5. If a given substance is kept a constant temperature, then the rate of change of its volume V with respect to its pressure P is the derivative dV/dP. The compressibility is defined by

$$\text{isothermal compressibility} = \beta = -\frac{1}{V}\frac{dV}{dP}.$$

The volume V (in cubic meters) of a sample of air at 25°C was found to be related to the pressure P (in kilopascals) by the equation

$$V = \frac{5.3}{P}.$$

Determine the compressibility when $P = 50$ kPa.

Example 6. Let $n = f(t)$ be the number of individuals in an animal or plant population at time t. The average rate of growth during a time period is the average rate of change of the growth of the population over that time period, and the rate of growth at a given time is the limit of the average rate of growth.

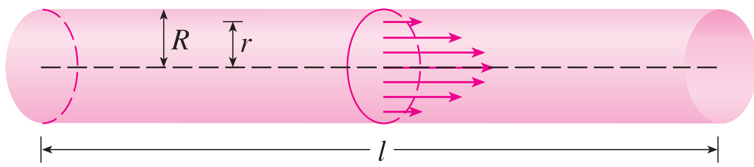
Suppose that a population of bacteria doubles every hour. The population function representing the bacteria's growth can be found to be

$$n = n_0 2^t$$

where n_0 is the initial population and the time t is measured in hours.

Find the rate of growth for a colony of bacteria with an initial population $n_0 = 100$ after 4 hours.

Example 7. The shape of a blood vessel can be modeled by a cylindrical tube with radius R and length l as illustrated in the figure.



The relationship between the velocity v of the blood and the distance r from the axis is given by the law of laminar flow

$$v = \frac{P}{4\eta l}(R^2 - r^2)$$

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P and l are constant, then v is a function of r with domain $[0, R]$. The velocity gradient at a given time is the limit of the average rate of change of the velocity.

For one of the smaller human arteries we can take $\eta = 0.027$, $R = 0.008$ cm, $l = 2$ cm, and $P = 4000$ dynes/cm^2. Find the speed at which blood is flowing at $r = 0.002$ and find the velocity gradient at that point.

Example 8. Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C is called a cost function. The instantaneous rate of change of cost with respect to the number of items produced, called the marginal cost, is the limit of the average rate of change of the cost.

Suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10,000 + 5x + 0.01x^2.$$

Find the marginal cost at the production level of 500 items and compare it to the actual cost of producing the 501st item.

3.8 Exponential Growth and Decay

Definition 3.8.1. The equation

$$\frac{dy}{dt} = ky$$

is called the law of natural growth (if $k > 0$) or the law of natural decay (if $k < 0$). It is called a differential equation because it involves an unknown function y and its derivative dy/dt.

Theorem 3.8.1. *The only solutions of the differential equation $dy/dt = ky$ are the exponential functions*

$$y(t) = y(0)e^{kt}.$$

Definition 3.8.2. If $P(t)$ is the size of a population at time t, then

$$k = \frac{1}{P}\frac{dP}{dt}$$

is the growth rate divided by population, called the relative growth rate.

Example 1. Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

Definition 3.8.3. If $m(t)$ is the mass remaining from an initial mass m_0 of a substance after time t, then the relative decay rate is

$$-\frac{1}{m}\frac{dm}{dt}.$$

It follows that the mass decays exponentially according to the equation

$$m(t) = m_0 e^{kt},$$

where the rate of decay is expressed in terms of half-life, the time required for half of any given quantity to decay.

Example 2. The half-life of radium-226 is 1590 years.

(a) A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of the sample that remains after t years.

(b) Find the mass after 1000 years correct to the nearest milligram.

(c) When will the mass be reduced to 30 mg?

Example 3. Newton's Law of Cooling can be represented as a differential equation
$$\frac{dT}{dt} = k(T - T_s),$$
where T is the temperature of the object at time t and T_s is the temperature of the surroundings. The exponential function $y(t) = y(0)e^{kt}$ is a solution to this differential equation when $y(t) = T(t) - T_s$.
A bottle of soda pop at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F. After half an hour the soda pop has cooled to 61°F.

(a) What is the temperature of the soda pop after another half hour?

(b) How long does it take for the soda pop to cool to 50°F?

Example 4. In general, if an amount A_0 is invested at an interest rate r, then after t years it is worth $A_0(1+r)^t$. Usually, however, interest is compounded more frequently, say, n times a year. Then in each compounding period the interest rate is r/n and there are nt compounding periods in t years, so the value of the investment is

$$A_0\left(1+\frac{r}{n}\right)^{nt}.$$

Therefore, taking limits gives us the amount after t years as

$$A(t) = A_0 e^{rt}$$

when interest is continuously compounded. Determine the value of an investment of \$1000 after 3 years of continuously compounding 6% interest. Compare this to the value of the same investment compounded annually instead.

3.9 Related Rates

Example 1. The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?

Example 2. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Example 3. Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

Example 4. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

Example 5. A plane flies horizontally at an altitude of 5 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$, this angle is decreasing at a rate of $\pi/6$ rad/min. How fast is the plane traveling at that time?

3.10 Linear Approximations and Differentials

Definition 3.10.1. The approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the linear approximation or tangent line approximation of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization of f at a.

Example 1. Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

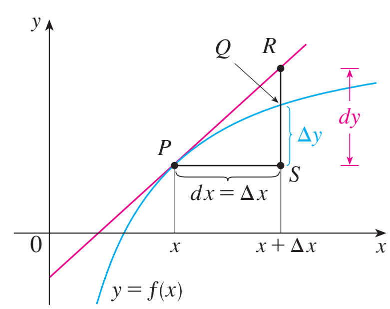
Example 2. For what values of x is the linear approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

accurate to within 0.5? What about accuracy to within 0.1?

Definition 3.10.2. If $y = f(x)$, where f is a differentiable function, then the differential dx is an independent variable; that is, dx can be given the value of any real number. The differential dy is then defined in terms of dx by the equation

$$dy = f'(x)dx.$$



Example 3. Compare the values Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes

(a) from 2 to 2.05

(b) from 2 to 2.01.

Example 4. The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

3.11 Hyperbolic Functions

Definition 3.11.1. Functions that have the same relationship to the hyperbola that trigonometric functions have to the circle are called hyperbolic functions and are defined as follows

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}.$$

Theorem 3.11.1 (Hyperbolic Identities)**.**

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$
$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$$

Example 1. Prove

(a) $\cosh^2 x - \sinh^2 x = 1$

(b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

Theorem 3.11.2 (Derivatives of Hyperbolic Functions)**.**

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$
$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x.$$

Example 2. Find $\dfrac{d}{dx}(\cosh\sqrt{x})$.

Theorem 3.11.3 (Inverse Hyperbolic Functions)**.**

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}) \qquad x \in \mathbb{R}$$
$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1}) \qquad x \geq 1$$
$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -1 < x < 1.$$

Example 3. Show that $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$.

Theorem 3.11.4 (Derivatives of Inverse Hyperbolic Functions)**.**

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}}$$
$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}.$$

Example 4. Prove that $\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$.

Example 5. Find $\dfrac{d}{dx}[\tanh^{-1}(\sin x)]$.

Chapter 4

Applications of Differentiation

4.1 Maximum and Minimum Values

Definition 4.1.1. Let c be a number in the domain D of a function f. Then $f(c)$ is the absolute maximum value (or global maximum value) of f on D if $f(c) \geq f(x)$ for all x in D and $f(c)$ is the absolute minimum value (or global minimum value) of f on D if $f(c) \leq f(x)$ for all x in D. These values are called extreme values of f.

Definition 4.1.2. The number $f(c)$ is a local maximum value of f if $f(c) \geq f(x)$ when x is near c and a local minimum value of f if $f(c) \leq f(x)$ when x is near c. When we say near, we mean on an open interval containing c. These values are called local extreme values of f.

Example 1. For what values of x does $f(x) = \sin x$ take on its maximum and minimum values?

Example 2. Find all of the extreme values of $f(x) = x^2$.

Example 3. Find all of the extreme values of $f(x) = x^3$.

Example 4. Find all of the extreme values of $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ within the domain $-2 \leq x \leq 3$.

Theorem 4.1.1 (Extreme Value Theorem)**.** *If f is continuous on a closed interval $[a, b]$ then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.*

Theorem 4.1.2 (Fermat's Theorem)**.** *If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.*

Proof. Suppose f has a local maximum at c. Then, by definition, $f(c) \geq f(x)$ if x is near c, so if we let $h > 0$ be close to 0 we have

$$\begin{aligned} f(c) &\geq f(c+h) \\ f(c+h) - f(c) &\leq 0 \\ \frac{f(c+h) - f(c)}{h} &\leq \frac{0}{h} \\ \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} &\leq \lim_{h \to 0^+} 0 \\ f'(c) &\leq 0. \end{aligned}$$

If $h < 0$, the direction of the inequality is reversed and we get $f'(c) \geq 0$. Thus combining these inequalities gives us $f'(c) = 0$. A similar argument can be used to achieve the same result if f has a local minimum at c. □

Example 5. Use the function $f(x) = x^3$ to determine whether the converse of Fermat's theorem is true.

Example 6. Does Fermat's theorem apply to the function $f(x) = |x|$?

Definition 4.1.3. A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 7. Find the critical numbers of $x^{1/3}(4 - x)^{2/3}$.

Example 8. Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 6x^2 + 5 \qquad -3 \leq x \leq 5.$$

Example 9. (a) Use a graphing device to estimate the absolute minimum and maximum values of the function $f(x) = x - 2\cos x$, $-2 \leq x \leq 0$.

(b) Use calculus to find the exact minimum and maximum values.

Example 10. The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery*. A model for the velocity of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ seconds, is given by

$$v(t) = 0.001302t^3 - 0.09029t^2 + 23.61t - 3.083$$

(in feet per second). Using this model, estimate the absolute maximum and minimum values of the acceleration of the shuttle between liftoff and the jettisoning of the boosters.

4.2 The Mean Value Theorem

Theorem 4.2.1 (Rolle's Theorem)**.** *Let f be a function that satisfies the following three hypotheses:*

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*
3. *$f(a) = f(b)$.*

Then there is a number c in (a, b) such that $f'(c) = 0$.

Proof. If $f(x) = k$, a constant, then $f'(x) = 0$ for all $x \in (a, b)$. If $f(x) > f(a)$ for some $x \in (a, b)$ then f has a local maximum for a number $c \in (a, b)$ by the extreme value theorem. Since f is differentiable on (a, b), $f'(c) = 0$ by Fermat's theorem. By the same reasoning, $f'(c) = 0$ if $f(x) < f(a)$. □

Example 1. How could Rolle's theorem be applied to a position function that models a ball thrown upward?

Example 2. Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Theorem 4.2.2 (The Mean Value Theorem)**.** *Let f be a function that satisfies the following hypotheses:*

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Let h be the difference between f and the secant line to f on $[a, b]$, i.e.,

$$h(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right].$$

Then h is continuous on $[a, b]$ and differentiable on (a, b) because it is the sum of f and a first-degree polynomial, which are both continuous on $[a, b]$ and differentiable on (a, b). Also,

$$\begin{aligned} h(a) &= f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0 \\ h(b) &= f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = 0, \end{aligned}$$

so $h(a) = h(b)$. Therefore, by Rolle's thereom, there is a number c in (a, b) such that $h'(c) = 0$, i.e.,

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a},$$

which is equivalent to

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

as desired. $\square$

Example 3. Find a number c in $(0, 2)$ such that the slope of the secant line is equal to the slope of the tangent line for the function $f(x) = x^3 - x$.

Example 4. What does the mean value theorem say about the velocity of an object moving in a straight line?

Example 5. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

Theorem 4.2.3. *If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).*

Proof. Let $x_1, x_2 \in (a, b)$ be such that $x_1 < x_2$. By the mean value theorem for f on $[x_1, x_2]$, we get

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1),$$

for some $c \in (x_1, x_2)$. But $f'(x) = 0$ for all x in this interval, so $f(x_2) = f(x_1)$. Since x_1 and x_2 were chosen arbitrarily, f is constant on (a, b). $\square$

Corollary 4.2.1. *If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is $f(x) = g(x) + c$ where c is a constant.*

Proof. Let
$$F(x) = f(x) - g(x).$$
Then
$$F'(x) = f'(x) - g'(x) = 0,$$
so F is constant by the previous theorem, and thus $f - g$ is constant. $\square$

Example 6. Prove the identity $\tan^{-1} x + \cot^{-1} x = \pi/2$.

4.3 Derivatives and the Shape of a Graph

Theorem 4.3.1 (Increasing/Decreasing Test)**.**

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Proof. Let x_1, x_2 be two numbers on an interval where $f'(x) > 0$ such that $x_1 < x_2$. Then by the mean value theorem,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c in the interval. But $f'(c) > 0$ and $x_2 - x_1 > 0$, so $f(x_2) - f(x_1) > 0$, i.e.,

$$f(x_2) > f(x_1)$$

in the interval. Since x_1 and x_2 were chosen arbitrarily, we are done, and the second half of the theorem is proved similarly. $\square$

Example 1. Find where the function $f(x) = 2x^3 - 15x^2 + 24x - 5$ is increasing and where it is decreasing.

Theorem 4.3.2 (The First Derivative Test)**.** *Suppose that c is a critical number of a continuous function f.*

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and to the right of c, or negative to the left and to the right of c, then f has no local minimum or maximum at c.

Example 2. Find the local minimum and maximum values of the function f in Example 1.

Example 3. Find the local maximum and minimum values of the function

$$g(x) = \sin x + \cos x \qquad 0 \leq x \leq 2\pi.$$

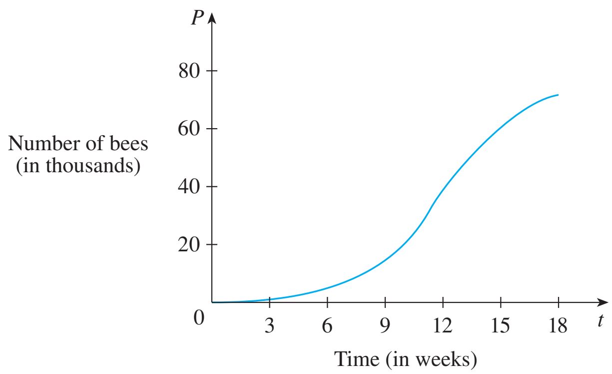
Definition 4.3.1. If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I. If the graph of f lies below all of its tangents on I, it is called concave downward on I.

Theorem 4.3.3 (Concavity Test)**.**

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Example 4. The figure shows a population graph for Cyprian honeybees raised in an apiary. How does the rate of population increase change over time? When is this rate highest? Over what intervals is P concave upward or concave downward?



Definition 4.3.2. A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Example 5. Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f'(x) > 0$ if $x \neq 2$, $f''(x) > 0$ if $x < 2$.

(ii) $f''(x) < 0$ if $x > 2$, f has an inflection point at $(2, 5)$.

(iii) $\lim_{x\to\infty} f(x) = 8$, $\lim_{x\to-\infty} f(x) = 0$.

Theorem 4.3.4 (The Second Derivative Test)**.** *Suppose f' is continuous near c.*

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example 6. Discuss the curve $y = 3x^4 - 8x^3 + 12$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Example 7. Sketch the graph of the function $f(x) = x^{1/3}(x+4)$.

Example 8. Use the first and second derivatives of $f(x) = e^{-2/x}$, together with asymptotes, to sketch its graph.

4.4 Indeterminate Forms and l'Hospital's Rule

Theorem 4.4.1 (L'Hospital's Rule)**.** *Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that*

$$\lim_{x\to a} f(x) = 0 \qquad \textit{and} \qquad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \qquad \textit{and} \qquad \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Example 1. Find $\displaystyle\lim_{x\to 1} \frac{\ln x}{x-1}$.

Example 2. Calculate $\displaystyle\lim_{x\to \infty} \frac{1+e^x}{\sqrt{x}}$.

Example 3. Calculate $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^{2/3}}$.

Example 4. Find $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x^3}$.

Example 5. Find $\lim_{x \to \pi^-} \dfrac{\sin x}{1 - \cos x}$.

Example 6. Evaluate $\lim_{x \to 0} \sin 5x \csc 3x$.

Example 7. Compute $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$.

Example 8. Calculate $\lim_{x \to \infty} (e^x - x)$.

Example 9. Calculate $\lim_{x\to 0^+}(4x+1)^{\cot x}$.

Example 10. Find $\lim_{x\to 0^+} x^{\sqrt{x}}$.

4.5 Summary of Curve Sketching

Use the following guidelines when sketching curves by hand:

A. Domain

B. Intercepts

C. Symmetry

D. Asymptotes

E. Intervals of Increase or Decrease

F. Local Maximum and Minimum Values

G. Concavity and Points of Inflection

Example 1. Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

Example 2. Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

Example 3. Sketch the graph of $f(x) = xe^x$.

Example 4. Sketch the graph of $f(x) = \dfrac{\cos x}{2 + \sin x}$.

Example 5. Sketch the graph of $y = \ln(4 - x^2)$.

Definition 4.5.1. If

$$\lim_{x\to\infty}[f(x) - (mx + b)] = 0$$

where $m \neq 0$, then the line $y = mx + b$ is called a slant asymptote because the vertical distance between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0.

Example 6. Sketch the graph of $f(x) = \dfrac{x^3}{x^2 + 1}$.

4.6 Graphing with Calculus and Calculators

Example 1. Graph the polynomial $f(x) = 2x^6 + 3x^5 + 3x^3 - 2x^2$. Use the graphs of f' and f'' to estimate all maximum and minimum points and intervals of concavity.

Example 2. Draw the graph of the function

$$f(x) = \frac{x^2 + 7x + 3}{x^2}$$

in a viewing rectangle that contains all the important features of the function. Estimate the maximum and minimum values and the intervals of concavity. Then use calculus to find these quantities exactly.

Example 3. Graph the function $f(x) = \dfrac{x^2(x+1)^3}{(x-2)^2(x-4)^4}$.

Example 4. Graph the function $f(x) = \sin(x + \sin 2x)$. For $0 \leq x \leq \pi$, estimate all maximum and minimum values, intervals of increase and decrease, and inflection points.

Example 5. How does the graph of $f(x) = 1/(x^2 + 2x + c)$ vary as c varies?

4.7 Optimization Problems

Example 1. A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Example 2. A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Theorem 4.7.1 (First Derivative Test for Absolute Extreme Values)**.** *Suppose that c is a critical number of a continuous function f defined on an interval.*

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Example 3. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

Example 4. An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east of the refinery. The cost of laying pipe is \$400,000/km over land to a point P on the north bank and \$800,000/km under the river to the tanks. To minimize the cost of the pipeline, where should P be located?

Example 5. Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Definition 4.7.1. If $p(x)$ is the price per unit that a company can charge if it sells x units, then p is called the demand function (or price function).
If x units are sold, then the total revenue

$$R(x) = \text{quantity} \times \text{price} = xp(x)$$

and R is called the revenue function. The derivative R' of the revenue function is called the marginal revenue function and is the rate of change of revenue with respect to the number of units sold.
If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

where C is the cost function and P is called the profit function. The marginal profit function is P', the derivative of the profit function.

Example 6. A baseball team plays in a stadium that seats 55,000 spectators. With ticket prices at \$10, the average attendance had been 27,000. When ticket prices were lowered to \$8, the average attendance rose to 33,000. Find the demand function, assuming that it is linear, and the revenue function. How should ticket prices be set to maximize revenue?

4.8 Newton's Method

Theorem 4.8.1 (Newton's Method)**.** *If x_n is the nth approximation of a root r for a function f then*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example 1. Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

Example 2. Use Newton's method to find $\sqrt[6]{2}$ to eight decimal places.

Example 3. Find, correct to six decimal places, the root of the equation $\cos x = x$.

4.9 Antiderivatives

Definition 4.9.1. A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem 4.9.1. *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is*

$$F(x) + C$$

where C is an arbitrary constant.

Proof. Follows by Corollary 4.2.1 to the mean value theorem. □

Example 1. Find the most general antiderivative of each of the following functions.

(a) $f(x) = \sin x$

(b) $f(x) = 1/x$

(c) $f(x) = x^n$, $n \neq -1$

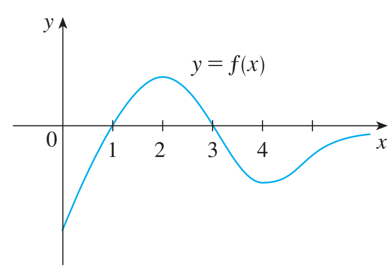
Example 2. Find all functions g such that

$$g'(x) = 2\cos x + \frac{2x - 4 + 3\sqrt{x}}{\sqrt{x}}.$$

Example 3. Find f if $f'(x) = e^x + 20(1 + x^2)^{-1}$ and $f(0) = -2$.

Example 4. Find f if $f''(x) = 4 + 6x + 24x^2$, $f(0) = 3$, and $f(1) = 10$.

Example 5. The graph of a function f is given in the figure. Make a rough sketch of an antiderivative F, given that $F(0) = 2$.



Example 6. A particle moves in a straight line and has acceleration given by $a(t) = 2t+1$. Its initial velocity is $v(0) = -2$ cm/s and its initial displacement is $s(0) = 3$ cm. Find its position function $s(t)$.

Example 7. A ball is thrown upward with a speed of 24 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground? [For motion close to the ground we may assume that the downward acceleration g is constant, its value being about 9.8 m/s^2 (or 32 ft/s^2).]

Chapter 10

Parametric Equations and Polar Coordinates

10.1 Curves Defined by Parametric Equations

Definition 10.1.1. Suppose that x and y are both given as functions of a third variable t (called a parameter) by the equations

$$x = f(t) \qquad y = g(t)$$

(called parametric equations). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a parametric curve.

Example 1. Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1.$$

Definition 10.1.2. In general, the curve with parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \leq t \leq b$$

has initial point $(f(a), g(a))$ and terminal point $(f(b), g(b))$.

Example 2. What curve is represented by the following parametric equations?

$$x = \cos t \qquad y = \sin t \qquad 0 \leq t \leq 2\pi.$$

Example 3. What curve is represented by the given parametric equations?

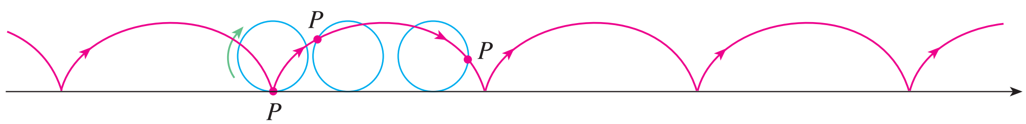
$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leq t \leq 2\pi.$$

Example 4. Find parametric equations for the circle with center (h, k) and radius r.

Example 5. Sketch the curve with parametric equations $x = \sin t$, $y = \sin^2 t$.

Example 6. Use a graphing device to graph the curve $x = y^4 - 3y^2$.

Example 7. The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a cycloid (see the figure). If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.



Example 8. Investigate the family of curves with parametric equations

$$x = a + \cos t \qquad y = a \tan t + \sin t.$$

What do these curves have in common? How does the shape change as a increases?

10.2 Calculus with Parametric Curves

Theorem 10.2.1. *Suppose f and g are differentiable functions. Then for a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x, we have*

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \textit{if } \frac{dx}{dt} \neq 0.$$

Proof. Since y is a differentiable function of x, we have, by the Chain Rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Then if $\frac{dx}{dt} \neq 0$ we can divide by it, so

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

□

Theorem 10.2.2. *Suppose f and g are differentiable functions. Then for a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x, we have*

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \qquad \textit{if } \frac{dx}{dt} \neq 0.$$

Proof. By the previous theorem,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \qquad \text{if } \frac{dx}{dt} \neq 0.$$

□

Example 1. A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

(a) Show that C has two tangents at the point $(3, 0)$ and find their equations

(b) Find the points on C where the tangent is horizontal or vertical.

(c) Determine where the curve is concave upward or downward.

(d) Sketch the curve.

Example 2.

(a) Find the tangent to the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ at the point where $\theta = \pi/3$.

(b) At what points is the tangent horizontal? When is it vertical?

Theorem 10.2.3. *If a curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then the area under the curve is given by*

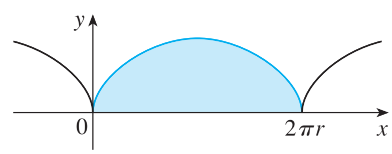
$$A = \int_{\alpha}^{\beta} g(t) f'(t)\, dt \qquad \left[\text{or } \int_{\beta}^{\alpha} g(t) f'(t)\, dt \right].$$

Proof. Since the area under the curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\, dx$, we can use the Substitution Rule for Definite Integrals with $y = g(t)$ and $dx = f'(t)\, dt$ to get

$$A = \int_a^b y\, dx = \int_{\alpha}^{\beta} g(t) f'(t)\, dt. \qquad \square$$

Example 3. Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta).$$



(See the figure.)

Theorem 10.2.4. *If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is*

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example 4. (a) Use the representation of the unit circle given by

$$x = \cos t \qquad y = \sin t \qquad 0 \leq t \leq 2\pi$$

to find its arc length.

(b) Use the representation of the unit circle given by

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leq t \leq 2\pi$$

to find its arc length.

Example 5. Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

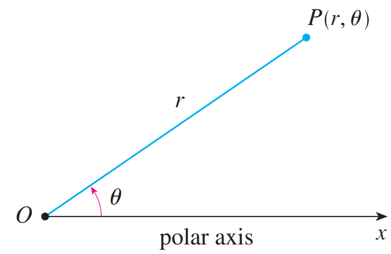
Theorem 10.2.5. *Suppose a curve C is given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f', g' are continuous, $g'(t) \geq 0$, is rotated about the x-axis. If C is traversed exactly once as t increases from α to β, then the area of the resulting surface is given by*

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example 6. Show that the surface area of a sphere of radius r is $4\pi r^2$.

10.3 Polar Coordinates

Definition 10.3.1. The polar coordinate system consists of a point called the pole (or origin) O, a ray starting at the pole called the polar axis, and other points P represented by (r, θ) where r is the distance from O to P and θ is the angle (usually measured in radians) between the polar axis and the line OP as in the figure. r, θ are called polar coordinates of P.



Example 1. Plot the points whose polar coordinates are given.

(a) $(1, 5\pi/4)$

(b) $(2, 3\pi)$

(c) $(2, -2\pi/3)$

(d) $(-3, 3\pi/4)$

Theorem 10.3.1. *If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then*

$$x = r\cos\theta \qquad y = r\sin\theta$$

and

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}.$$

Example 2. Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

Example 3. Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

Example 4. What curve is represented by the polar equation $r = 2$?

Example 5. Sketch the polar curve $\theta = 1$.

Example 6. (a) Sketch the curve with polar equation $r = 2\cos\theta$.

(b) Find a Cartesian equation for this curve.

Example 7. Sketch the curve $r = 1 + \sin\theta$.

Example 8. Sketch the curve $r = \cos 2\theta$.

Theorem 10.3.2. *The slope of the tangent line to a polar curve $r = f(\theta)$ is*

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Proof. Regard θ as a parameter and write

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Then by Theorem 10.2.1 and the product rule, we have

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

□

Example 9.

(a) For the cardioid $r = 1 + \sin\theta$ of Example 7, find the slope of the tangent line when $\theta = \pi/3$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

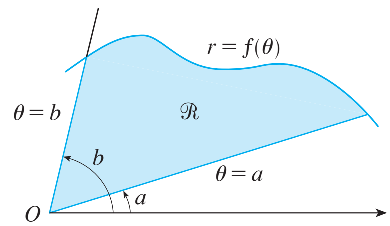
Example 10. Graph the curve $r = \sin(8\theta/5)$.

Example 11. Investigate the family of polar curves given by $r = 1 + c\sin\theta$. How does the shape change as c changes? (These curves are called limaçons, after a French word for snail, because of the shape of the curves for certain values of c.)

10.4 Areas and Lengths in Polar Coordinates

Theorem 10.4.1. *Let $\mathscr{R}$ be the region, illustrated in the figure, bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leq 2\pi$. The area A of the polar region $\mathscr{R}$ is*

$$A = \int_a^b \frac{1}{2} r^2 \, d\theta.$$



Example 1. Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Example 2. Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Example 3. Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

Theorem 10.4.2. *The length of a curve with polar equation $r = f(\theta)$, $a \leq \theta \leq b$, is*

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Proof. Regard θ as a parameter and write

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Then by the product rule, we have

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \qquad \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta.$$

Since $\cos^2\theta + \sin^2\theta = 1$,

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2,\end{aligned}$$

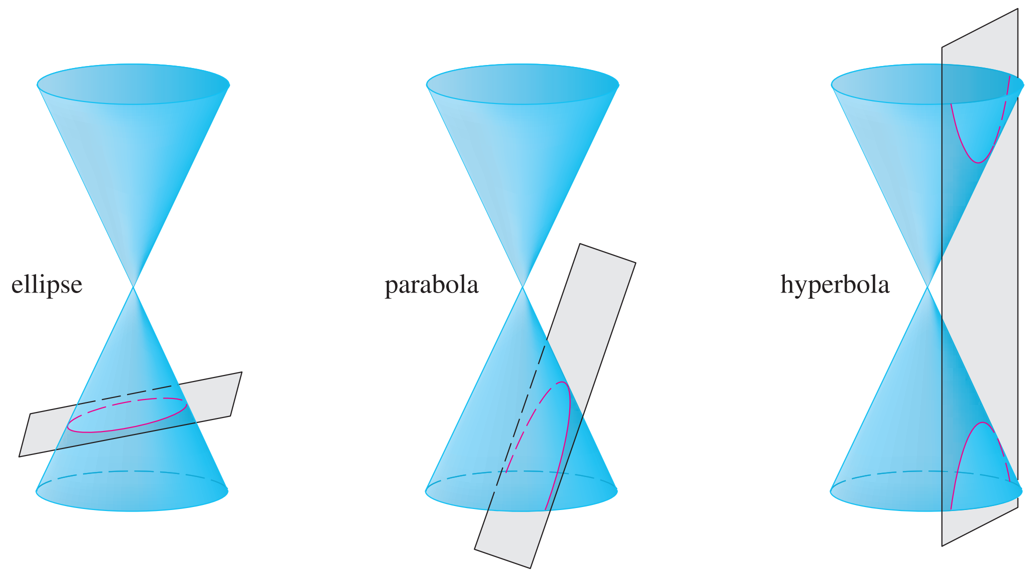
so

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \qquad \square$$

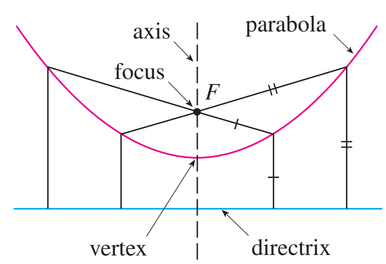
Example 4. Find the length of the cardioid $r = 1 + \sin\theta$.

10.5 Conic Sections

Definition 10.5.1. Parabolas, ellipses, and hyperbolas are called conic sections, or conics, because they result from intersecting a cone with a plane as shown in the figure.



Definition 10.5.2. A parabola is the set of points in a plane that are equidistant from a fixed point F (called the focus) and a fixed line (called the directrix). This definition is illustrated by the figure. Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the vertex. The line through the focus perpendicular to the directrix is called the axis of the parabola.



Theorem 10.5.1. *An equation of the parabola with focus* $(0, p)$ *and directrix* $y = -p$ *is*

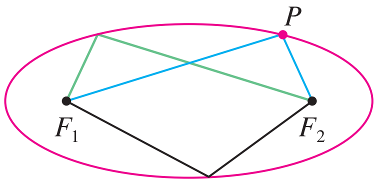
$$x^2 = 4py.$$

Theorem 10.5.2. *An equation of the parabola with focus* $(p, 0)$ *and directrix* $x = -p$ *is*

$$y^2 = 4px.$$

Example 1. Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

Definition 10.5.3. An ellipse is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant (see the figure). These two fixed points are called the foci (plural of focus).



Definition 10.5.4. If $(-c, 0)$ and $(c, 0)$ are the foci of an ellipse, the sum of the distances from a point on the ellipse to the foci are $2a > 0$, and $b^2 = a^2 - c^2$, then the points $(a, 0)$ and $(-a, 0)$ are called the vertices of ellipse and the line segment joining the vertices is called the major axis. The line segment joining $(0, b)$ and $(0, -b)$ is the minor axis.

Theorem 10.5.3. *The ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \geq b > 0$$

has foci $(\pm c, 0)$, *where* $c^2 = a^2 - b^2$, *and vertices* $(\pm a, 0)$.

Theorem 10.5.4. *The ellipse*

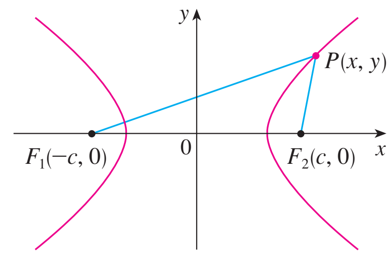
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \geq b > 0$$

has foci $(0, \pm c)$, *where* $c^2 = a^2 - b^2$, *and vertices* $(0, \pm a)$.

Example 2. Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

Example 3. Find an equation of the ellipse with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.

Definition 10.5.5. A hyperbola is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the foci) is a constant. This definition is illustrated in the figure.



Theorem 10.5.5. *The hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci $(\pm c, 0)$, *where* $c^2 = a^2 + b^2$, *vertices* $(\pm a, 0)$, *and asymptotes* $y = \pm(b/a)x$.

Theorem 10.5.6. *The hyperbola*

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has foci $(0, \pm c)$, *where* $c^2 = a^2 + b^2$, *vertices* $(0, \pm a)$, *and asymptotes* $y = \pm(a/b)x$.

Example 4. Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

Example 5. Find the foci and equation of the hyperbola with vertices $(0, \pm 1)$ and asymptote $y = 2x$.

Example 6. Find an equation of the ellipse with foci $(2, -2)$, $(4, -2)$, and vertices $(1, -2)$, $(5, -2)$.

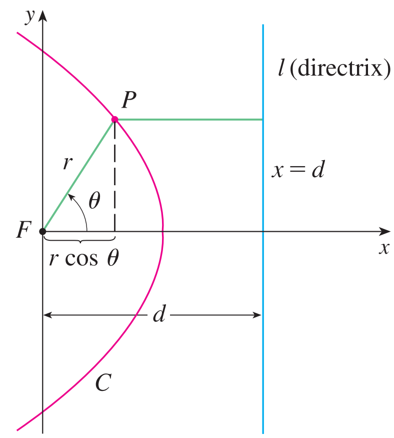
Example 7. Sketch the conic $9x^2 - 4y^2 - 72x + 8y + 176 = 0$ and find its foci.

10.6 Conic Sections in Polar Coordinates

Theorem 10.6.1. *Let F be a fixed point (called the focus) and l be a fixed line (called the directrix) in a plane. Let e be a fixed positive number (called the eccentricity). The set of all points P in the plane such that*

$$\frac{|PF|}{|Pl|} = e$$

(that is, the ratio of the distance from F to the distance from l is the constant e) is a conic section. The conic is



(a) an ellipse if $e < 1$

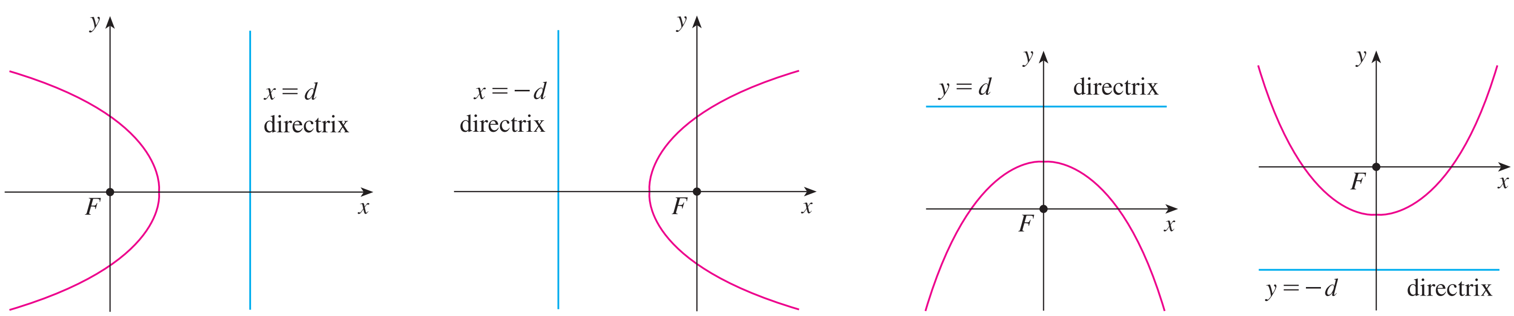
(b) a parabola if $e = 1$

(c) a hyperbola if $e > 1$

Theorem 10.6.2. *A polar equation of the form*

$$r = \frac{ed}{1 \pm e\cos\theta} \qquad or \qquad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.



(a) $r = \dfrac{ed}{1 + e\cos\theta}$ (b) $r = \dfrac{ed}{1 - e\cos\theta}$ (c) $r = \dfrac{ed}{1 + e\sin\theta}$ (d) $r = \dfrac{ed}{1 - e\sin\theta}$

Example 1. Find a polar equation for a parabola that has its focus at the origin and whose directrix is the line $y = -6$.

Example 2. A conic is given by the polar equation

$$r = \frac{10}{3 - 2\cos\theta}.$$

Find the eccentricity, identify the conic, locate the directrix, and sketch the conic.

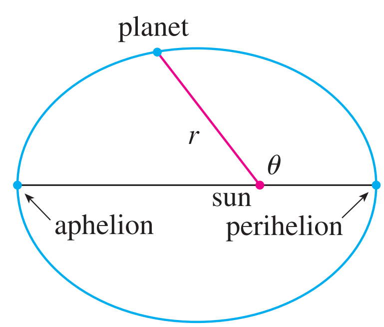
Example 3. Sketch the conic $r = \dfrac{12}{2 + 4\sin\theta}$.

Example 4. If the ellipse of Example 2 is rotated through an angle $\pi/4$ about the origin, find a polar equation and graph the resulting ellipse.

Theorem 10.6.3. *The polar equation of an ellipse with focus at the origin, semimajor axis a, eccentricity e, and directrix $x = d$ can be written in the form*

$$r = \frac{a(1-e^2)}{1+e\cos\theta}.$$

Definition 10.6.1. The positions of a planet that are closest to and farthest from the sun are called its perihelion and aphelion, respectively, and correspond to the vertices of the ellipse (see the figure). The distances from the sun to the perihelion and aphelion are called the perihelion distance and aphelion distance, respectively.



Theorem 10.6.4. *The perihelion distance from a planet to the sun is $a(1-e)$ and the aphelion distance is $a(1+e)$.*

Proof. If the sun is at the focus F, at perihelion we have $\theta = 0$, so

$$r = \frac{a(1-e^2)}{1+e\cos 0} = \frac{a(1-e)(1+e)}{1+e} = a(1-e).$$

Similarly, at aphelion $\theta = \pi$ and $r = a(1+e)$. $\square$

Example 5. (a) Find an approximate polar equation for the elliptical orbit of the earth around the sun (at one focus) given that the eccentricity is about 0.017 and the length of the major axis is about 2.99×10^8 km.

(b) Find the distance from the earth to the sun at perihelion and at aphelion.

Index

Bibliography

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