Calculus II Notes

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Contents

Chapter 5

Integrals

5.1 Areas and Distances

Example 1. Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

Example 2. For the region in Example 1, show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is,

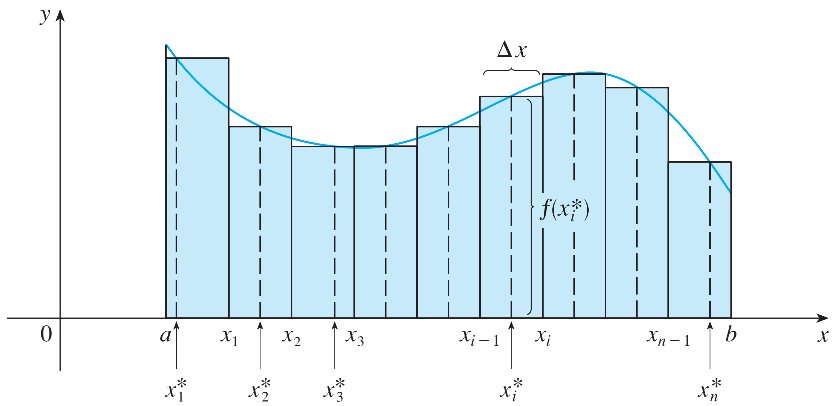
$$\lim_{n\to\infty} R_n = \frac{1}{3}.$$

Definition 5.1.1. The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

The last equality is an example of the use of sigma notation to write sums with many terms more compactly.

Definition 5.1.2. Numbers x_i^* in the ith subinterval $[x_{i-1}, x_i]$ are called sample points. We form lower (and upper) sums by choosing the sample points x_i^* so that $f(x_i^*)$ is the minimum (and maximum) value of f on the ith subinterval.



Example 3. Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

(b) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

Example 4. Suppose the odometer on a car is broken. Estimate the distance driven in feet over a 30-second time interval by using the speedometer readings taken every five seconds and recorded in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

5.2 The Definite Integral

Definition 5.2.1. If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0(= a), x_1, x_2, \ldots, x_n(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the <u>definite integral of f from a to b</u> is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

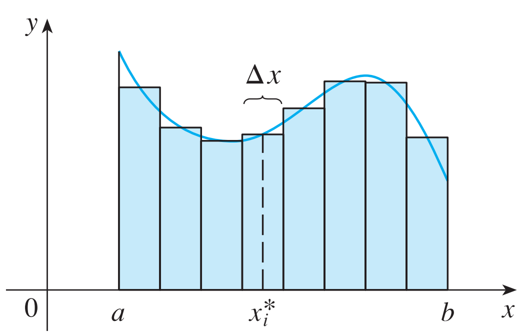
provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is <u>integrable</u> on $[a, b]$.

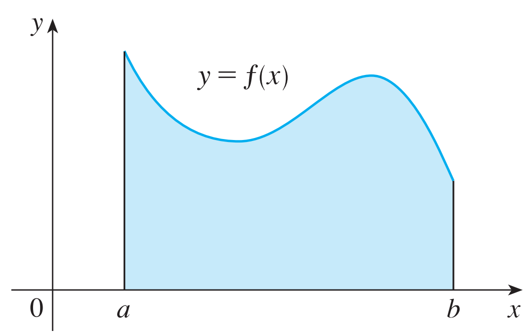
Definition 5.2.2. The symbol $\int$ is called an <u>integral sign</u>. In the notation $\int_a^b f(x)dx$, $f(x)$ is called the <u>integrand</u> and a and b are called the <u>limits of integration</u>; a is the <u>lower limit</u> and b is the <u>upper limit</u>. The procedure of calculating an integral is called <u>integration</u>.

Definition 5.2.3. The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

is called a <u>Riemann sum</u> and it can be used to approximate the definite integral of an integrable function within any desired degree of accuracy.

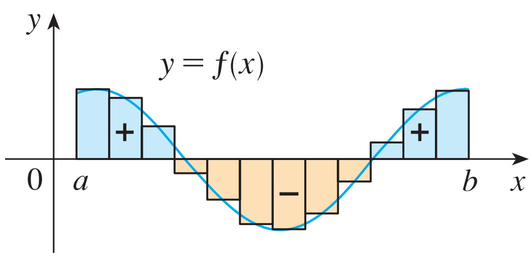


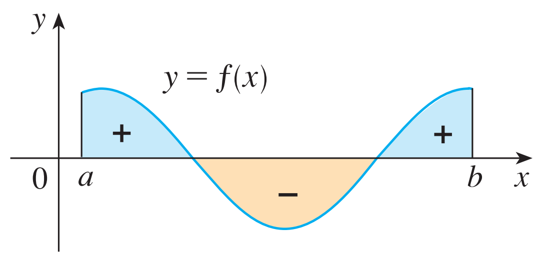


Definition 5.2.4. A definite integral can be interpreted as a net area, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and the above the graph of f.





Theorem 5.2.1. *If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.*

Theorem 5.2.2. *If f is integrable on $[a, b]$, then*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n} \qquad \textit{and} \qquad x_i = a + i\Delta x.$$

Example 1. Express

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x$$

as an integral on the interval $[0, \pi]$.

Theorem 5.2.3. *The following formulas are true when working with sigma notation:*

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i.$$

Example 2. (a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

(b) Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

Example 3. (a) Set up an expression for $\int_1^3 e^x \, dx$ as a limit of sums.

(b) Use a computer algebra system to evaluate the expression.

Example 4. Evaluate the following integrals by interpreting each in terms of areas.

(a) $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$

(b) $\displaystyle\int_0^3 (x-1)\,dx$

Theorem 5.2.4 (Midpoint Rule)**.**

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x[f(\bar{x}_1) + \cdots + f(\bar{x}_n)]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \textit{midpoint of } [x_{i-1}, x_i].$$

Example 5. Use the Midpoint Rule with $n = 5$ to approximate $\displaystyle\int_1^2 \frac{1}{x}\,dx$.

Theorem 5.2.5 (Properties of the Definite Integral)**.**

1. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

2. $\displaystyle\int_a^a f(x)\,dx = 0.$

3. $\displaystyle\int_a^b c\,dx = c(b-a),$ *where c is any constant.*

4. $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

5. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx,$ *where c is any constant.*

6. $\displaystyle\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$

7. $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$

Example 6. Use the properties of integrals to evaluate $\displaystyle\int_0^1 (4+3x^2)\,dx.$

Example 7. If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^{8} f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$.

Theorem 5.2.6 (Comparison Properties of the Integral)**.**

8. *If* $f(x) \geq 0$ *for* $a \leq x \leq b$*, then* $\int_a^b f(x)\,dx \geq 0$.

9. *If* $f(x) \geq g(x)$ *for* $a \leq x \leq b$*, then* $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

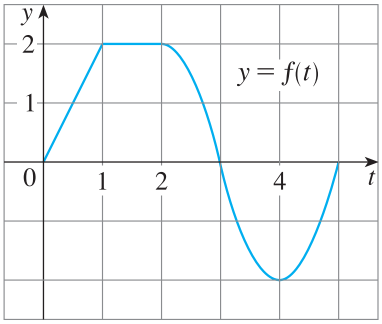
10. *If* $m \leq f(x) \leq M$ *for* $a \leq x \leq b$*, then*

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Example 8. Use Property 10 to estimate $\int_0^1 e^{-x^2}\,dx$.

5.3 The Fundamental Theorem of Calculus

Example 1. If f is the function whose graph is shown in the figure and $g(x) = \int_0^x f(t)\,dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$. Then sketch a rough graph of g.



Theorem 5.3.1 (The Fundamental Theorem of Calculus, Part 1)**.** *If f is continuous on $[a, b]$, then the function g defined by*

$$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

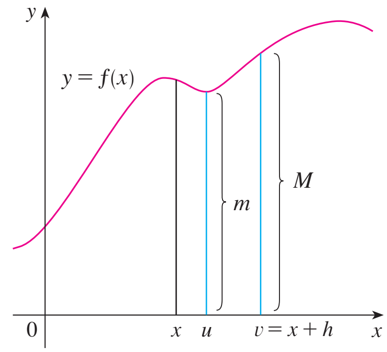
is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Proof. If x and $x + h$ are in (a, b), then

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \left(\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt \right) - \int_a^x f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt \end{aligned}$$

and so, for $h \neq 0$,

$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$



For now let's assume that $h > 0$. Since f is continuous on $[x, x+h]$, the Extreme Value Theorem says that there are numbers u and v in $[x, x+h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x, x+h]$. (See the figure.)
Then

$$\begin{aligned} mh &\leq \int_x^{x+h} f(t)\,dt \leq Mh \\ f(u)h &\leq \int_x^{x+h} f(t)\,dt \leq f(v)h \\ f(u) &\leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq f(v) \\ f(u) &\leq \frac{g(x+h)-g(x)}{h} \leq f(v). \end{aligned}$$

This inequality can be proved in a similar manner for the case where $h < 0$. Now we let $h \to 0$. Then $u \to x$ and $v \to x$, since u and v lie between x and $x+h$. Therefore

$$\lim_{h\to 0} f(u) = \lim_{u\to x} f(u) = f(x) \qquad \text{and} \qquad \lim_{h\to 0} f(v) = \lim_{u\to x} f(v) = f(x)$$

because f is continuous at x. We conclude, from the Squeeze Theorem, that

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x).$$

If $x = a$ or b, then this equation can be interpreted as a one-sided limit, and thus g is continuous on $[a, b]$. □

Example 2. Find the derivative of the function $g(x) = \displaystyle\int_0^x \sqrt{1+t^2}\,dt$.

Example 3. Find the derivative of the Fresnel function

$$S(x) = \int_0^x \sin(\pi t^2/2)\, dt$$

and compare its graph with that of $S(x)$ to visually confirm the fundamental theorem of calculus.

Example 4. Find $\dfrac{d}{dx} \displaystyle\int_1^{x^4} \sec t \, dt$.

Theorem 5.3.2 (The Fundamental Theorem of Calculus, Part 2)**.** *If f is continuous on $[a,b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Proof. Let $g(x) = \int_a^x f(t)\,dt$. By Part 1, $g'(x) = f(x)$; that is, g is an antiderivative of f. If F is any other antiderivative of f on $[a,b]$, then, by Corollary 4.2.1,

$$F(x) = g(x) + C$$

for $a < x < b$. By continuity, this is also true for $x \in [a,b]$, so again by Part 1,

$$g(a) = \int_a^a f(t)\,dt = 0$$

and thus

$$\begin{aligned} F(b) - F(a) &= [g(b) + C] - [g(a) + C] \\ &= g(b) + C - 0 - C \\ &= g(b) \\ &= \int_a^b f(t)\,dt. \end{aligned}$$

□

Example 5. Evaluate the integral $\displaystyle\int_1^3 e^x\,dx$.

Remark 1. We often use the notation

$$F(x)\Big]_a^b = F(b) - F(a).$$

So the equation of the Fundamental Theorem of Calculus Part 2 can be written as

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \qquad \text{where} \qquad F' = f.$$

Other common notations are $F(x)|_a^b$ and $[F(x)]_a^b$.

Example 6. Find the area under the parabola $y = x^2$ from 0 to 1.

Example 7. Evaluate $\displaystyle\int_3^6 \frac{dx}{x}$.

Example 8. Find the area under the cosine curve from 0 to b, where $0 \le b \le \pi/2$.

Example 9. What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \left.\frac{x^{-1}}{-1}\right]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

5.4 Indefinite Integrals and the Net Change Theorem

Definition 5.4.1. An antiderivative of f is called an indefinite integral where

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x).$$

Example 1. Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\,dx.$$

Example 2. Evaluate $\displaystyle\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$.

Example 3. Evaluate $\displaystyle\int_0^3 (x^3 - 6x)\,dx$.

Example 4. Find $\displaystyle\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx$ and interpret the result in terms of areas.

Example 5. Evaluate $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt$.

Theorem 5.4.1 (Net Change Theorem). *The integral of a rate of change is the net change:*

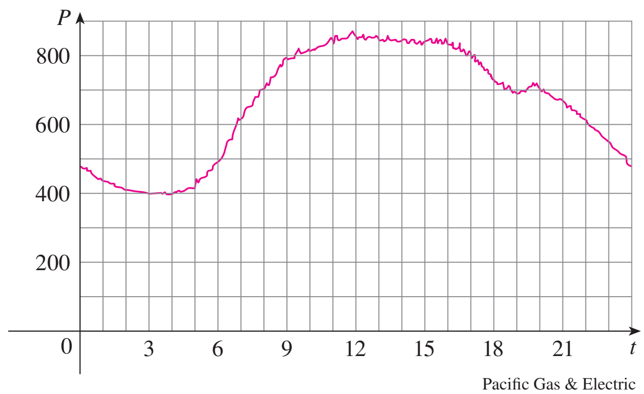
$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Example 6. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

(b) Find the distance traveled during this time period.

Example 7. The figure shows the power consumption in the city of San Francisco for a day in September (P is measured in megawatts; t is measured in hours starting at midnight). Estimate the energy used on that day.



5.5 The Substitution Rule

Theorem 5.5.1 (The Substitution Rule)**.** *If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Proof. If $f = F'$, then, by the Chain Rule,

$$\frac{d}{dx}[F(g(x))] = f(g(x))g'(x).$$

Thus if $u = g(x)$, then we have

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C = F(u) + C = \int f(u)\,du.$$

□

Example 1. Find $\int x^3 \cos(x^4 + 2)\,dx$.

Example 2. Evaluate $\int \sqrt{2x+1}\,dx$.

Example 3. Find $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

Example 4. Calculate $\displaystyle\int e^{5x}\,dx$.

Example 5. Find $\displaystyle\int \sqrt{1+x^2}x^5\,dx$.

Example 6. Calculate $\int \tan x\, dx$.

Theorem 5.5.2 (The Substitution Rule for Definite Integrals)**.** *If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then*

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Proof. Let F be an antiderivative of f. Then $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$, so by part 2 of the fundamental theorem of calculus, we have

$$\int_a^b f(g(x))g'(x)\, dx = F(g(x))\big]_a^b = F(g(b)) - F(g(a)).$$

By applying part 2 a second time, we also have

$$\int_{g(a)}^{g(b)} f(u)\, du = F(u)\big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a)).$$ □

Example 7. Evaluate $\int_0^4 \sqrt{2x+1}\, dx$.

Example 8. Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$.

Example 9. Calculate $\int_1^e \frac{\ln x}{x}\, dx$.

Theorem 5.5.3 (Integrals of Symmetric Functions)**.** *Suppose f is continuous on $[-a, a]$.*

(a) If f is even $[f(-x) = f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$

(b) If f is odd $[f(-x) = -f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$

Proof. First we split the integral:

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

By substituting $u = -x$ we get $du = -dx$ and $u = a$ when $x = -a$, so

$$-\int_{0}^{-a} f(x)\,dx = -\int_{0}^{a} f(-u)\,(-du) = \int_{0}^{a} f(-u)\,du$$

and therefore

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx.$$

(a) If f is even then $f(-u) = f(u)$, so

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

(b) If f is odd then $f(-u) = -f(u)$, so

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 0. \qquad \square$$

Example 10. Evaluate $\displaystyle\int_{-2}^{2} (x^6 + 1)\, dx$.

Example 11. Evaluate $\displaystyle\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\, dx$.

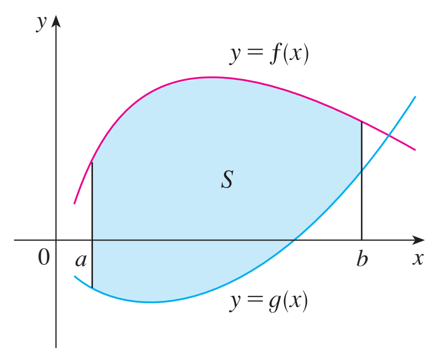
Chapter 6

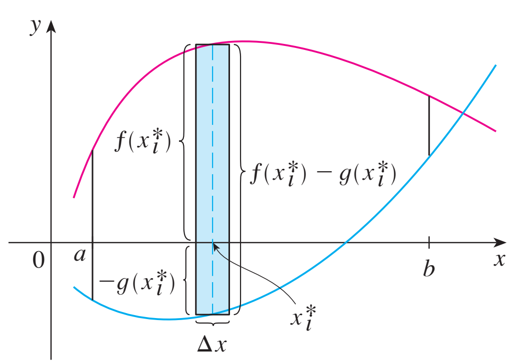
Applications of Integration

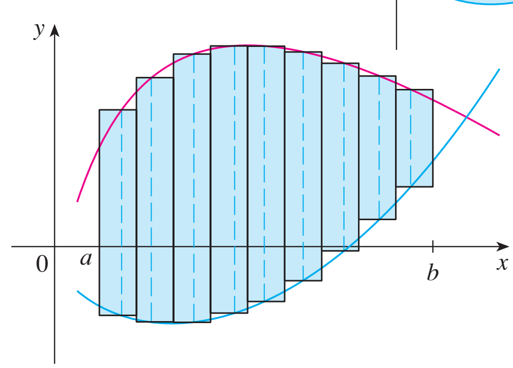
6.1 Areas Between Curves

Definition 6.1.1. The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x = \int_a^b [f(x) - g(x)]\, dx.$$





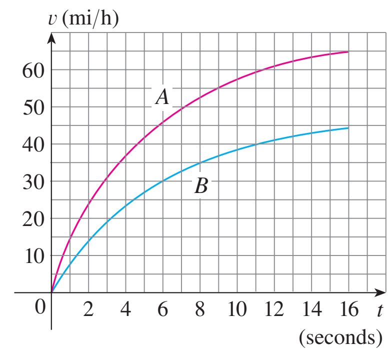


Example 1. Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

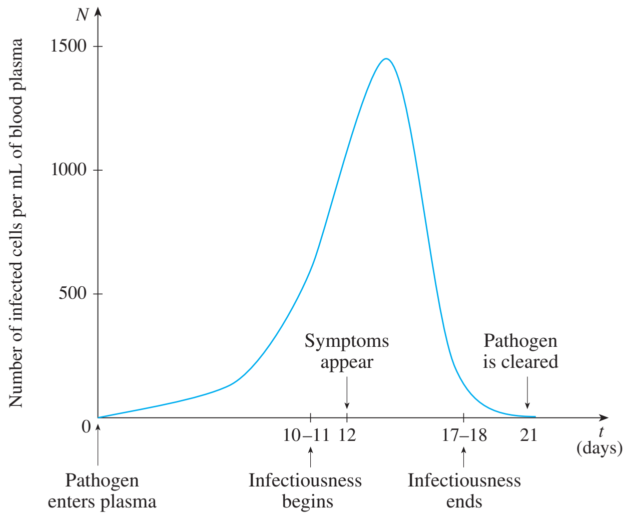
Example 2. Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Example 3. Find the approximate area of the region bounded by the curves $y = x/\sqrt{x^2+1}$ and $y = x^4 - x$.

Example 4. The figure shows the velocity curves for two cars, A and B, that start side by side and move along the same road. What does the area between the curves represent? Use the Midpoint Rule to estimate it.



Example 5. The figure is an example of a pathogenesis curve for a measles infection. It shows how the disease develops in an individual with no immunity after the measles virus spreads to the bloodstream from the respiratory tract.



The patient becomes infectious to others once the concentration of infected cells becomes great enough, and he or she remains infectious until the immune system manages to prevent further transmission. However, symptoms don't develop until the "amount of infection" reaches a particular threshold. The amount of infection needed to develop symptoms depends on both the concentration of infected cells and time, and corresponds to the area under the pathogenesis curve until symptoms appear.

(a) The pathogenesis curve in the figure has been modeled by $f(t) = -t(t-21)(t+1)$. If infectiousness begins on day $t_1 = 10$ and ends on day $t_2 = 18$, what are the corresponding concentration levels of infected cells?

(b) The level of infectiousness for an infected person is the area between $N = f(t)$ and the line through the points $P_1(t_1, (f(t_1))$ and $P_2(t_2, f(t_2))$, measured in (cells/mL)$\cdot$ days. Compute the level of infectiousness for this particular patient.

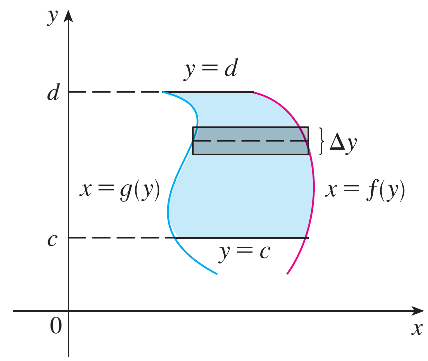
Definition 6.1.2. The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

Example 6. Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

Remark 1. Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ (see the figure), then its area is

$$A = \int_c^d [f(y) - g(y)]\, dy.$$

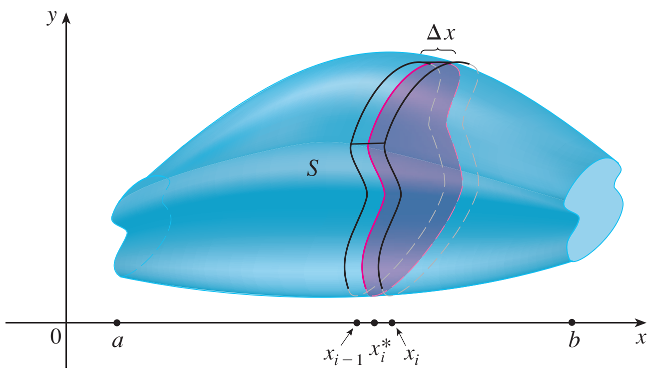


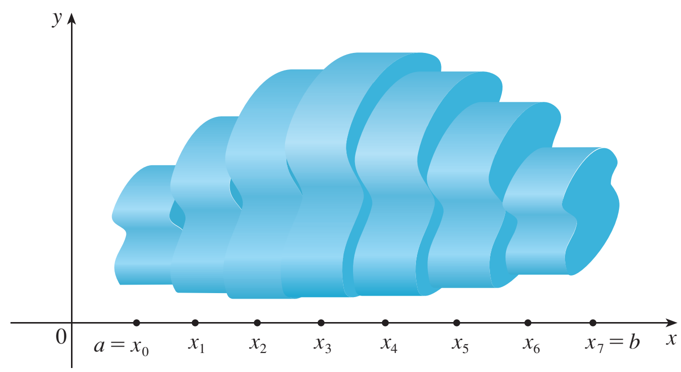
Example 7. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

6.2 Volumes

Definition 6.2.1 (Definition of Volume)**.** Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx.$$





Example 1. Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Example 2. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.

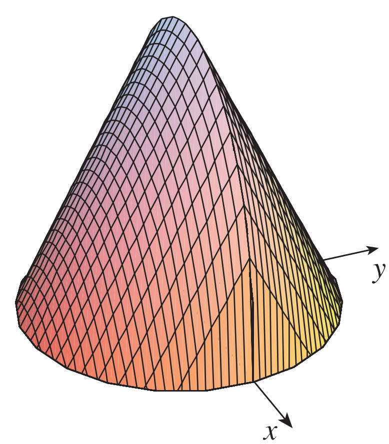
Example 3. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

Example 4. The region $\mathscr{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

Example 5. Find the volume of the solid obtained by rotating the region in Example 4 about the line $y = 2$.

Example 6. Find the volume of the solid obtained by rotating the region in Example 4 about the line $x = -1$.

Example 7. The figure shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.



Example 8. Find the volume of a pyramid whose base is a square with side L and whose height is h.

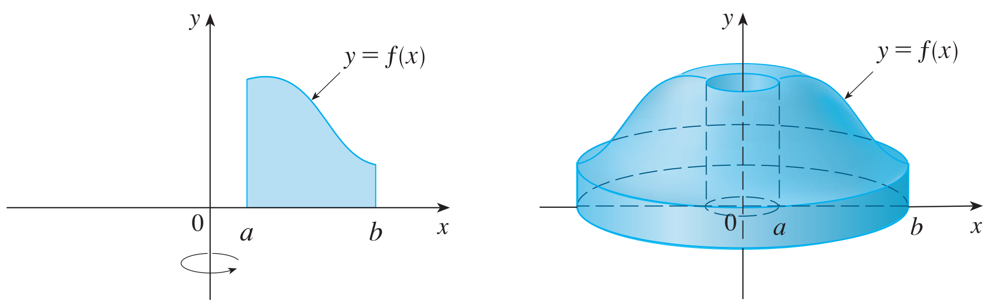
Example 9. A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of $30°$ along a diameter of the cylinder. Find the volume of the wedge.

6.3 Volumes by Cylindrical Shells

Theorem 6.3.1 (Method of Cylindrical Shells)**.** *The volume of the solid in the figure, obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is*

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\Delta x = \int_a^b 2\pi x f(x)\, dx \qquad \textit{where } 0 \le a \le b$$

and where $\bar{x}_i$ is the midpoint of the ith subinterval $[x_{i-1}, x_i]$.



Example 1. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Example 2. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

Example 3. Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

Example 4. Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

6.4 Work

Definition 6.4.1. In general, if an object moves along a straight line with position function $s(t)$, then the force F on the object (in the same direction) is given by Newton's Second Law of Motion as the product of its mass m and its acceleration a:

$$F = ma = m\frac{d^2s}{dt^2}.$$

Definition 6.4.2. In the case of constant acceleration, the force F is also constant and the work done is defined to be the product of the force F and distance d that the object moves:

$$W = Fd \qquad \text{work} = \text{force} \times \text{distance}.$$

Example 1. (a) How much work is done in lifting a 1.2-kg book off the floor to put it on a desk that is 0.7 m high? Use the fact that the acceleration due to gravity is $g = 9.8\ \text{m/s}^2$.

(b) How much work is done in lifting a 20-lb weight 6 ft off the ground?

Definition 6.4.3. If the force $f(x)$ on an object is variable, then we define the work done in moving the object from a to b as

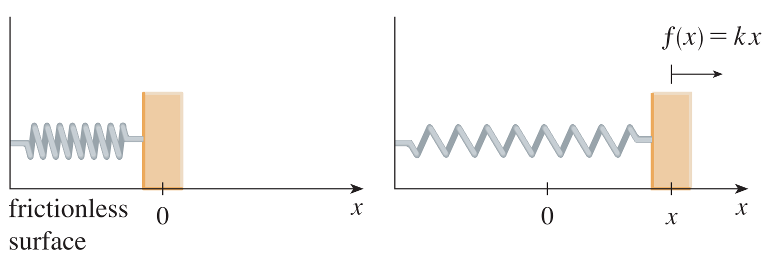
$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx.$$

Example 2. When a particle is located a distance x feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

Theorem 6.4.1 (Hooke's Law)**.** *The force required to maintain a spring stretched x units beyond its natural length is proportional to x:*

$$f(x) = kx$$

where k is a positive constant called the spring constant (see the figure). Hooke's Law holds provided that x is not too large.



(a) Natural position of spring (b) Stretched position of spring

Example 3. A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

Example 4. A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

Example 5. A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3.)

6.5 Average Value of a Function

Definition 6.5.1. The average value of a function f on the interval $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Example 1. Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

Theorem 6.5.1 (The Mean Value Theorem for Integrals)**.** *If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that*

$$f(c) = f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\, dx,$$

that is,

$$\int_a^b f(x)\, dx = f(c)(b-a).$$

Proof. By applying the Mean Value Theorem for derivatives to the function $F(x) = \int_a^x f(t)dt$, we see that there exists a number c in $[a, b]$ such that

$$\begin{aligned} F'(c) &= \frac{F(b) - F(a)}{b-a} \\ \frac{d}{dx}\left[\int_a^x f(t)\, dt\right]\Bigg|_c &= \frac{F(b) - F(a)}{b-a} \\ f(c) &= \frac{1}{b-a}[F(b) - F(a)] \\ &= \frac{1}{b-a} \int_a^b f(x)\, dx. \end{aligned}$$ □

Example 2. Find a number c in the interval $[-1, 2]$ that satisfies the mean value theorem for integrals for the function $f(x) = 1 + x^2$.

Example 3. Show that the average velocity of a car over a time interval $[t_1, t_2]$ is the same as the average of its velocities during the trip.

Chapter 7

Techniques of Integration

7.1 Integration by Parts

Theorem 7.1.1 (Formula for Integration by Parts)**.** *If f and g are differentiable functions then*

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx,$$

or, equivalently,

$$\int u\,dv = uv - \int v\,du$$

where $u = f(x)$ and $v = g(x)$.

Proof. By the Product Rule,

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= f(x)g'(x) + g(x)f'(x) \\
f(x)g(x) &= \int [f(x)g'(x) + g(x)f'(x)]\,dx \\
&= \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx \\
\int f(x)g'(x)\,dx &= f(x)g(x) - \int g(x)f'(x)\,dx
\end{aligned}$$

□

Example 1. Find $\displaystyle\int x \sin x \, dx$.

Example 2. Evaluate $\displaystyle\int \ln x \, dx$.

Example 3. Find $\int t^2 e^t \, dt$.

Example 4. Evaluate $\int e^x \sin x \, dx$.

Theorem 7.1.2 (Formula for Definite Integration by Parts)**.** *If f and g are differentiable on (a, b) and f' and g' are continuous, then*

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx.$$

Example 5. Calculate $\int_0^1 \tan^{-1} x\,dx$.

Example 6. Prove the reduction formula

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

where $n \geq 2$ is an integer.

7.2 Trigonometric Integrals

Example 1. Evaluate $\displaystyle\int \cos^3 x \, dx$.

Example 2. Find $\displaystyle\int \sin^5 x \cos^2 x \, dx$.

Remark 1. Sometimes it is easier to use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \text{and} \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

to evaluate an integral.

Example 3. Evaluate $\displaystyle\int_0^\pi \sin^2 x \, dx$.

Example 4. Find $\displaystyle\int \sin^4 x \, dx$.

Example 5. Evaluate $\displaystyle\int \tan^6 x \sec^4 x \, dx$.

Example 6. Find $\displaystyle\int \tan^5 \theta \sec^7 \theta \, d\theta$.

Example 7. Find $\displaystyle\int \tan^3 x\,dx$.

Example 8. Find $\displaystyle\int \sec^3 x\,dx$.

Remark 2. To evaluate the integrals (a) $\int \sin mx \cos nx \, dx$, (b) $\int \sin mx \sin nx \, dx$, or (c) $\int \cos mx \cos nx \, dx$, use the corresponding identity:

$$\text{(a) } \sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$
$$\text{(b) } \sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\text{(c) } \cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)].$$

Example 9. Evaluate $\displaystyle\int \sin 4x \cos 5x \, dx$.

7.3 Trigonometric Substitution

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ -\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ -\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\ 0 \leq \theta \leq \dfrac{\pi}{2}$ or $\pi \leq \theta \leq \dfrac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Example 1. Evaluate $\displaystyle\int \frac{\sqrt{9 - x^2}}{x^2}\,dx$.

Example 2. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Example 3. Find $\int \frac{1}{x^2\sqrt{x^2+4}}\, dx$.

Example 4. Find $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\,dx$.

Example 5. Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$, where $a > 0$.

Example 6. Find $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx.$

Example 7. Evaluate $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$.

7.4 Integration by Partial Fractions

Example 1. Find $\displaystyle\int \frac{x^3+x}{x-1}\,dx$.

Example 2. Evaluate $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$.

Example 3. Find $\displaystyle\int \frac{dx}{x^2 - a^2}$, where $a \neq 0$.

Example 4. Find $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx.$

Theorem 7.4.1.

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C.$$

Example 5. Evaluate $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$.

Example 6. Evaluate $\displaystyle\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\, dx$.

Example 7. Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}.$$

Example 8. Evaluate $\displaystyle\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\,dx.$

Example 9. Evaluate $\displaystyle\int \frac{\sqrt{x+4}}{x}\,dx$.

7.5 Strategy for Integration

Example 1. $\displaystyle\int \frac{\tan^3 x}{\cos^3 x}\,dx.$

Example 2. $\displaystyle\int e^{\sqrt{x}}\,dx.$

Example 3. $\displaystyle\int \frac{x^5+1}{x^3-3x^2-10x}\,dx.$

Example 4. $\displaystyle\int \frac{dx}{x\sqrt{\ln x}}$.

Example 5. $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\,dx$.

7.6 Integration Using Tables and CAS's

Example 1. The region bounded by the curves $y = \arctan x$, $y = 0$, and $x = 1$ is rotated about the y-axis. Find the volume of the resulting solid.

Example 2. Use the Table of Integrals to find $\displaystyle\int \frac{x^2}{\sqrt{5 - 4x^2}}\, dx$.

Example 3. Use the Table of Integrals to evaluate $\int x^3 \sin x \, dx$.

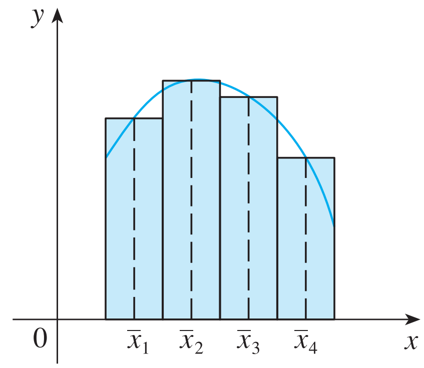
Example 4. Use the Table of Integrals to find $\int x\sqrt{x^2 + 2x + 4}\, dx$.

Example 5. Use a computer algebra system to find $\int x\sqrt{x^2+2x+4}\,dx$.

Example 6. Use a CAS to evaluate $\int x(x^2+5)^8\,dx$.

Example 7. Use a CAS to find $\int \sin^5 x\cos^2 x\,dx$.

7.7 Approximate Integration



Theorem 7.7.1 (Midpoint Rule)**.**

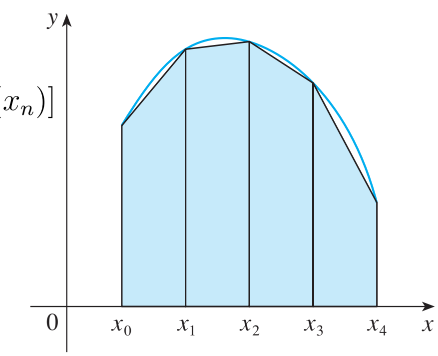
$$\int_a^b f(x)\,dx \approx M_n = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \textit{midpoint of } [x_{i-1}, x_i].$$



Theorem 7.7.2 (Trapezoidal Rule)**.**

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)]$$

where $\Delta x = (b-a)/n$ *and* $x_i = a + i\Delta x$.

Example 1. Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with $n = 5$ to approximate the integral $\int_1^2 (1/x)\,dx$.

Theorem 7.7.3 (Error Bounds). *Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then*

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad \textit{and} \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2}.$$

Example 2. How large should we take n in order to guarantee that the Trapezoidal and Midpoint Rule approximations for $\int_1^2 (1/x)\, dx$ are accurate to within 0.0001?

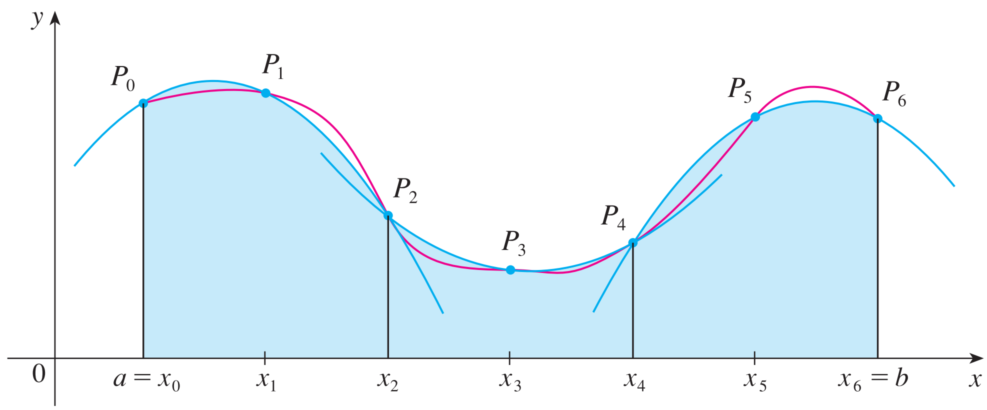
Example 3. (a) Use the Midpoint Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.

(b) Give an upper bound for the error involved in this approximation.

Theorem 7.7.4 (Simpson's Rule)**.**

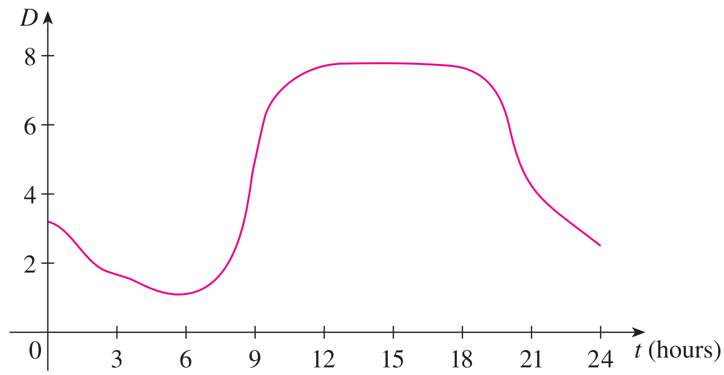
$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n is even and $\Delta x = (b-a)/n$.



Example 4. Use Simpson's Rule with $n = 10$ to approximate $\int_1^2 (1/x)\,dx$.

Example 5. The figure shows data traffic on the link from the United States to SWITCH, the Swiss academic and research network, on February 10, 1998. $D(t)$ is the data throughput, measured in megabits per second (Mb/s). Use Simpson's Rule to estimate the total amount of data transmitted on the link from midnight to noon on that day.



Theorem 7.7.5 (Error Bound for Simpson's Rule)**.** *Suppose that* $|f^{(4)}(x)| \leq K$ *for* $a \leq x \leq b$*. If* E_S *is the error involved in using Simpson's Rule, then*

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

Example 6. How large should we take n in order to guarantee that the Simpson's Rule approximation for $\int_1^2 (1/x)\,dx$ is accurate to within 0.0001?

Example 7. (a) Use Simpson's Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.

(b) Estimate the error involved in this approximation.

7.8 Improper Integrals

Definition 7.8.1 (Definition of an Improper Integral of Type 1)**.**

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

(c) If both $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx.$$

In part (c) any real number a can be used.

Example 1. Determine whether the integral $\int_1^{\infty} (1/x)\,dx$ is convergent or divergent.

Example 2. Evaluate $\int_{-\infty}^{0} xe^x \, dx$.

Example 3. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Example 4. For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

convergent?

Definition 7.8.2 (Definition of an Improper Integral of Type 2)**.**

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called <u>convergent</u> if the corresponding limit exists and <u>divergent</u> if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example 5. Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

Example 6. Determine whether $\displaystyle\int_0^{\pi/2} \sec x \, dx$ converges or diverges.

Example 7. Evaluate $\displaystyle\int_0^3 \frac{dx}{x-1}$ if possible.

Example 8. $\displaystyle\int_0^1 \ln x\, dx.$

Theorem 7.8.1 (Comparison Theorem)**.** *Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.*

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Example 9. Show that $\displaystyle\int_0^\infty e^{-x^2}\,dx$ is convergent.

Example 10. Determine whether $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ converges or diverges.

Chapter 11

Infinite Sequences and Series

11.1 Sequences

Definition 11.1.1. A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots.$$

The number a_1 is called the first term, a_2 is the second term, and in general a_n is the nth term.
A sequence can also be defined as a function whose domain is the set of positive integers. However, we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.
The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}.$$

Example 1. Some sequences can be defined by giving a formula for the nth term. In the following examples we give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that n doesn't have to start at 1.

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$	$a_n = \dfrac{n}{n+1}$	$\left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dots, \dfrac{n}{n+1}, \dots\right\}$
(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}$	$a_n = \dfrac{(-1)^n(n+1)}{3^n}$	$\left\{-\dfrac{2}{3}, \dfrac{3}{9}, -\dfrac{4}{27}, \dfrac{5}{81}, \dots, \dfrac{(-1)^n(n+1)}{3^n}, \dots\right\}$
(c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$	$a_n = \sqrt{n-3}, n \geq 3$	$\left\{0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots\right\}$
(d) $\left\{\cos\dfrac{n\pi}{6}\right\}_{n=0}^{\infty}$	$a_n = \cos\dfrac{n\pi}{6}, n \geq 0$	$\left\{1, \dfrac{\sqrt{3}}{2}, \dfrac{1}{2}, 0, \dots, \cos\dfrac{n\pi}{6}, \dots\right\}$

Example 2. Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \dots\right\}$$

assuming that the pattern of the first few terms continues.

Example 3. Here are some sequences that don't have a simple defining equation.

(a) The sequence $\{p_n\}$, where p_n is the population of the world as of January 1 in the year n.

(b) If we let a_n be the digit in the nth decimal place of the number e, then $\{a_n\}$ is a well-defined sequence whose first few terms are

$$\{7, 1, 8, 2, 8, 1, 8, 2, 4, 5, \ldots\}.$$

(c) The Fibonacci sequence $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1 \qquad f_2 = 1 \qquad f_n = f_{n-1} + f_{n-2} \qquad n \geq 3.$$

Each term is the sum of the two preceding terms. The first few terms are

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits.

Definition 11.1.2. A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty}$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

Definition 11.1.3 (Precise Definition of the Limit of a Sequence)**.** A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon.$$

Theorem 11.1.1. *If* $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = a_n$ *when* n *is an integer, then* $\lim_{n\to\infty} a_n = L$.

Definition 11.1.4. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad a_n > M.$$

Theorem 11.1.2 (Limit Laws for Sequences)**.** *If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then*

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty}(a_n-b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$
$$\lim_{n\to\infty}(a_nb_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \textit{if } \lim_{n\to\infty} b_n \neq 0$$
$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \textit{if } p>0 \textit{ and } a_n>0.$$

Theorem 11.1.3 (Squeeze Theorem for Sequences)**.** *If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

Theorem 11.1.4. *If $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Proof. Since $\lim_{n\to\infty}|a_n| = 0$,

$$\lim_{n\to\infty} -|a_n| = 0 = -\lim_{n\to\infty}|a_n| = 0.$$

But $-|a_n| \le a_n \le |a_n|$ for all n, so by the squeeze theorem for sequences, $\lim_{n\to\infty} a_n = 0$. □

Example 4. Find $\lim_{n\to\infty}\frac{n}{n+1}$.

Example 5. Is the sequence $a_n = \dfrac{n}{\sqrt{10+n}}$ convergent or divergent?

Example 6. Calculate $\lim\limits_{n\to\infty} \dfrac{\ln n}{n}$.

Example 7. Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Example 8. Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

Theorem 11.1.5. *If* $\lim_{n\to\infty} a_n = L$ *and the function* f *is continuous at* L*, then*

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Proof. Choose a particular n, say n_0. By the definition of a limit of a sequence, given $\varepsilon_1 > 0$ there exists an integer N, such that $|a_{n_0} - L| < \varepsilon_1$ for $n_0 > N$. Similarly, by the definition of continuity, the limit of f exists at L, so for $\varepsilon_2 > 0$ there exists $\varepsilon_1 > 0$ such that if $|a_{n_0} - L| < \varepsilon_1$ then $|f(a_{n_0}) - f(L)| < \varepsilon_2$. This is true for arbitrary $\varepsilon_2 > 0$, so $\lim_{n\to\infty} f(a_n) = f(L)$. $\square$

Example 9. Find $\lim_{n\to\infty} \sin(\pi/n)$.

Example 10. Discuss the convergence of the sequence $a_n = n!/n^n$, where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.

Example 11. For what values of r is the sequence $\{r^n\}$ convergent?

Definition 11.1.5. A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is monotonic if it is either increasing or decreasing.

Example 12. Is the sequence $\left\{ \dfrac{3}{n+5} \right\}$ increasing or decreasing?

Example 13. Show that the sequence $a_n = \dfrac{n}{n^2+1}$ is decreasing.

Definition 11.1.6. A sequence $\{a_n\}$ is bounded above if there is a number M such that

$$a_n \leq M \qquad \text{for all } n \geq 1.$$

It is bounded below if there is a number m such that

$$m \leq a_n \qquad \text{for all } n \geq 1.$$

If it is bounded above and below, then $\{a_n\}$ is a bounded sequence.

Theorem 11.1.6 (Monotonic Sequence theorem)**.** *Every bounded, monotonic sequence is convergent.*

Example 14. Investigate the sequence $\{a_n\}$ defined by the recurrence relation

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{2}(a_n + 6) \qquad \text{for } n = 1, 2, 3, \ldots.$$

11.2 Series

Definition 11.2.1. In general, if we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an infnite series (or just a series) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n.$$

Definition 11.2.2. Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} = s.$$

The number s is called the sum of the series. If the sequence $\{s_n\}$ is divergent, then the series is called divergent.

Example 1. Find the sum of the series $\sum_{n=1}^{\infty} a_n$ if the sum of the first n terms of the series is

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}.$$

Example 2. Find the sum of the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

where each term is obtained from the preceding one by multiplying it by the common ratio r.

Example 3. Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots .$$

Example 4. Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

Example 5. A drug is administered to a patient at the same time every day. Suppose the concentration of the drug is C_n (measured in mg/mL) after the injection on the nth day. Before the injection the next day, only 30% of the drug remains in the bloodstream and the daily dose raises the concentration by 0.2 mg/mL.

(a) Find the concentration after three days.

(b) What is the concentration after the nth dose?

(c) What is the limiting concentration?

Example 6. Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.

Example 7. Find the sum of the series $\sum\limits_{n=0}^{\infty} x^n$, where $|x| < 1$.

Example 8. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

Example 9. Show that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

Theorem 11.2.1. *If the series* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. Since $\sum a_n$ is convergent, the sequence $\{s_n\}$ is convergent. Let $\lim_{n\to\infty} s_n = s$. Since $n-1 \to \infty$ as $n \to \infty$, we also have $\lim_{n\to\infty} s_{n-1} = s$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0. \qquad \square$$

Corollary 11.2.1 (Test for Divergence)**.** *If* $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Proof. If the series is not divergent, then it is convergent, and so $\lim_{n\to\infty} a_n = 0$ by Theorem 11.2.1. The result follows by contrapositive. $\square$

Example 10. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Theorem 11.2.2. *If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$, and*

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Example 11. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.

Remark 1. A finite number of terms doesn't affect the convergence or divergence of a series. For instance, suppose that we were able to show that the series

$$\sum_{n=4}^{\infty} \frac{n}{n^3+1}$$

is convergent. Since

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \sum_{n=4}^{\infty} \frac{n}{n^3+1}$$

it follows that the entire series $\sum_{n=1}^{\infty} n/(n^3+1)$ is convergent. Similarly, if it is known that the series $\sum_{n=N+1}^{\infty} a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.

11.3 The Integral Test and Estimates of Sums

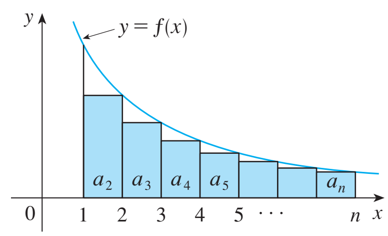
Theorem 11.3.1 (The Integral Test). *Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and $a_n = f(n)$. The the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:*

(i) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Proof.

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then comparing the areas of the rectangles with the area under $y = f(x)$ from 1 to n in the top figure, we see that



$$\sum_{i=2}^{n} a_i = a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,dx \le \int_1^{\infty} f(x)\,dx$$

since $f(x) \ge 0$. Therefore

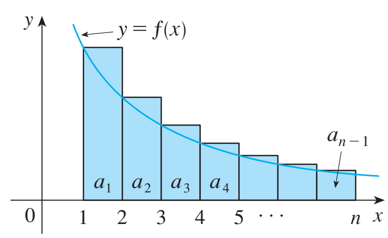
$$s_n = a_1 + \sum_{i=2}^{n} a_i \le a_1 + \int_1^{\infty} f(x)\,dx = M, \text{ say.}$$

Since $s_n \le M$ for all n, the sequence $\{s_n\}$ is bounded above. Also

$$s_{n+1} = s_n + a_{n+1} \ge s_n$$

since $a_{n+1} = f(n+1) \ge 0$. Thus $\{s_n\}$ is an increasing bounded sequence and so it is convergent by the Monotonic Sequence Theorem.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\int_1^n f(x)\,dx \to \infty$ as $n \to \infty$ because $f(x) \ge 0$. But the bottom figure shows that



$$\int_1^n f(x)\,dx \le a_1 + a_2 + \cdots + a_{n-1} = \sum_{i=1}^{n-1} a_i = s_{n-1}$$

and so $s_{n-1} \to \infty$, implying that $s_n \to \infty$. □

Example 1. Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Example 2. For what values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent? (This series is called the p-series.)

Example 3. Determine whether each series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$

Example 4. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

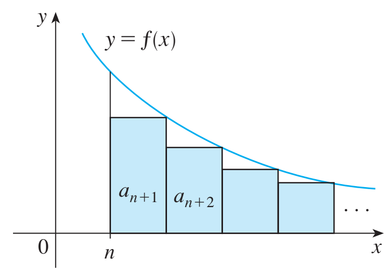
Definition 11.3.1. The <u>remainder</u>

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

is the error made when s_n, the sum of the first n terms, is used as an approximation to the total sum.

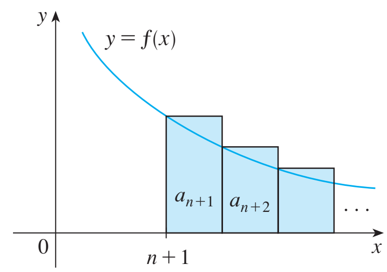
Theorem 11.3.2 (Remainder Estimate for the Integral Test)**.** *Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then*

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$



Proof. Comparing the rectangles with the area under $y = f(x)$ for $x > n$ in the top figure, we see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_{n}^{\infty} f(x)\,dx.$$



Similarly, we see from the bottom figure that

$$R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^{\infty} f(x)\,dx. \qquad \square$$

Example 5. (a) Approximate the sum of the series $\sum 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

Corollary 11.3.1. *Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. Then*

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\,dx.$$

Example 6. Use Corollary 11.3.1 with $n = 10$ to estimate the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

11.4 The Comparison Tests

Theorem 11.4.1 (The Comparison Test). *Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.*

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Proof. (i) Let

$$s_n = \sum_{i=1}^{n} a_i \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{\infty} b_n$$

Since both series have positive terms, the sequences $\{s_n\}$ and $\{t_n\}$ are increasing ($s_{n+1} = s_n + a_{n+1} \geq s_n$). Also $t_n \to t$, so $t_n \leq t$ for all n. Since $a_i \leq b_i$, we have $s_n \leq t_n$. Thus $s_n \leq t$ for all n. This means that $\{s_n\}$ is increasing and bounded above and therefore converges by the Monotonic Sequence Theorem. Thus $\sum a_n$ converges.

(ii) If $\sum b_n$ is divergent, then $t_n \to \infty$ (since $\{t_n\}$ is increasing). But $a_i \geq b_i$ so $s_n \geq t_n$. Thus $s_n \to \infty$. Therefore $\sum a_n$ diverges. $\square$

Example 1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

Example 2. Test the series $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ for convergence or divergence.

Theorem 11.4.2 (The Limit Comparison Test)**.** *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with positive terms. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c *is a finite number and* $c > 0$*, then either both series converge or both diverge.*

Proof. Let m and M be positive numbers such that $m < c < M$. Because a_n/b_n is close to c for large n, there is an integer N such that

$$m < \frac{a_n}{b_n} < M \qquad \text{when } n > N,$$

and so

$$mb_n < a_n < Mb_n \qquad \text{when } n > N.$$

If $\sum b_n$ converges, so does $\sum Mb_n$. Thus $\sum a_n$ converges by part (i) of the Comparison Test. If $\sum b_n$ diverges, so does $\sum mb_n$ and part (ii) of the Comparison Test shows that $\sum a_n$ diverges. $\square$

Example 3. Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

Example 4. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

Example 5. Use the sum of the first 100 terms to approximate the sum of the series $\sum 1/(n^3+1)$. Estimate the error involved in this approximation.

11.5 Alternating Series

Definition 11.5.1. An alternating series is a series whose terms are alternately positive and negative.

Theorem 11.5.1 (Alternating Series Test)**.** *If the alternating series*

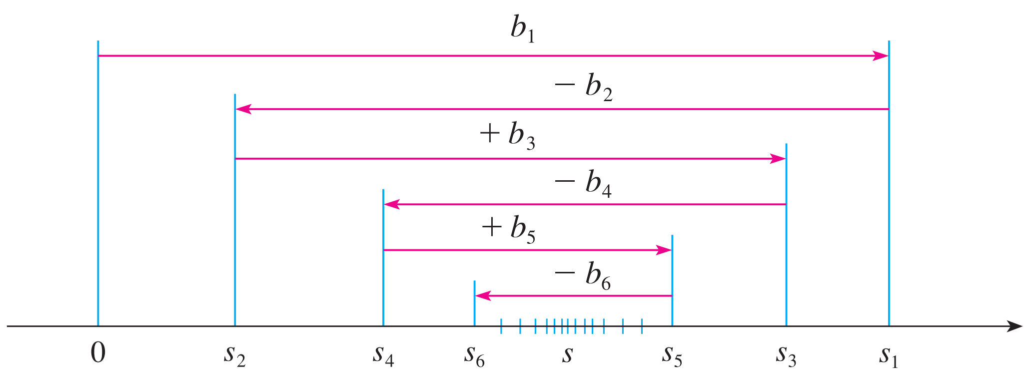
$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

$$\begin{aligned} &(i)\ \ b_{n+1} \le b_n \qquad \text{for all } n \\ &(ii)\ \ \lim_{n\to\infty} b_n = 0 \end{aligned}$$

then the series is convergent.

Proof.



We first consider the even partial sums:

$$\begin{aligned} s_2 &= b_1 - b_2 \ge 0 && \text{since } b_2 \le b_1 \\ s_4 &= s_2 + (b_3 - b_4) \ge s_2 && \text{since } b_4 \le b_3. \end{aligned}$$

In general

$$s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \ge s_{2n-2} \qquad \text{since } b_{2n} \le b_{2n-1}.$$

Thus

$$0 \le s_2 \le s_4 \le s_6 \le \cdots \le s_{2n} \le \cdots .$$

But we can also write

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}.$$

Every term in parenthesis is positive, so $s_{2n} \leq b_1$ for all n. Therefore, the sequence $\{s_{2n}\}$ of even partial sums is increasing and bounded above. It is therefore convergent by the Monotonic Sequence Theorem. Let's call its limit s, that is,

$$\lim_{n\to\infty} s_{2n} = s.$$

Now we compute the limit of the odd partial sums:

$$\begin{aligned}\lim_{n\to\infty} s_{2n+1} &= \lim_{n\to\infty} (s_{2n} + b_{2n+1}) \\ &= \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} \\ &= s + 0 \\ &= s.\end{aligned}$$

Since both the even and odd partial sums converge to s, we have $\lim_{n\to\infty} s_n = s$ and so the series is convergent. $\square$

Example 1. Determine whether the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

converges or diverges.

Example 2. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ converges or diverges.

Example 3. Test the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ for convergence or divergence.

Theorem 11.5.2 (Alternating Series Estimation Theorem)**.** *If* $s = \sum(-1)^{n-1}b_n$, *where* $b_n > 0$, *is the sum of an alternating series that satisfies*

$$\text{(i) } b_{n+1} \leq b_n \qquad \text{and} \qquad \text{(ii) } \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Proof. We know from the proof of the Alternating Series Test that s lies between any two consecutive partial sums s_n and s_{n+1}. (There we showed that s is larger than all the even partial sums. A similar argument shows that s is smaller than all the odd sums.) It follows that

$$|s - s_n| \leq |s_{n+1} - s_n| = b_{n+1}. \qquad \square$$

Example 4. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

11.6 Absolute Convergence, Ratio and Root Tests

Definition 11.6.1. A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

Example 1. Is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

absolutely convergent?

Example 2. Is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

absolutely convergent?

Definition 11.6.2. A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

Theorem 11.6.1. *If a series $\sum a_n$ is absolutely convergent, then it is convergent.*

Proof. Observe that the inequality

$$0 \le a_n + |a_n| \le 2|a_n|$$

is true because $|a_n|$ is either a_n or $-a_n$. If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent, so $\sum 2|a_n|$ is convergent. Therefore, by the Comparison Test, $\sum (a_n + |a_n|)$ is convergent. Then

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

is the difference of two convergent series and is therefore convergent. $\square$

Example 3. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

Theorem 11.6.2 (The Ratio Test)**.**

(i) *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent (and therefore convergent).*

(ii) *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ *or* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(iii) *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$*, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of* $\sum a_n$*.*

Example 4. Test the series $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ for absolute convergence.

Example 5. Test the convergence of the series $\sum_{n=1}^{\infty}\frac{n^n}{n!}$.

Theorem 11.6.3 (The Root Test)**.**

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent (and therefore convergent).*

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ *or* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$*, the Root Test is inconclusive.*

Example 6. Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

Definition 11.6.3. By a rearrangement of an infinite series $\sum a_n$ we mean a series obtained by simply changing the order of the terms.

Remark 1. If $\sum a_n$ is an absolutely convergent series with sum s, then any rearrangement of $\sum a_n$ has the same sum s.

Remark 2. If $\sum a_n$ is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\sum a_n$ that has a sum equal to r. For example, if we multiply the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2$$

by $\frac{1}{2}$, we get

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots = \frac{1}{2}\ln 2.$$

Then inserting zeros between the terms of this series gives

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots = \frac{1}{2}\ln 2,$$

and we can add this to the alternating harmonic series to get

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}\ln 2,$$

which is a rearrangement of the alternating harmonic series with a different sum.

11.7 Strategy for Testing Series

Example 1. $\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$.

Example 2. $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$.

Example 3. $\sum_{n=1}^{\infty} ne^{-n^2}$.

Example 4. $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{n^4+1}$.

Example 5. $\sum_{n=1}^{\infty}\frac{2^k}{k!}$.

Example 6. $\sum_{n=1}^{\infty}\frac{1}{2+3^n}$.

11.8 Power Series

Definition 11.8.1. A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.
More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a power series in $(x-a)$ or a power series centered at a or a power series about a.

Example 1. For what values of x is the series $\sum_{n=0}^{\infty} n! x^n$ convergent?

Example 2. For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Example 3. Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

Theorem 11.8.1. *For a given power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$*, there are only three possibilities:*

(i) The series converges only when $x = a$*.*

(ii) The series converges for all x*.*

(iii) There is a positive number R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$*.*

Definition 11.8.2. The number R in case (iii) is called the radius of convergence of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii). The interval of convergence of a power series is the interval that consists of all values of x for which the series converges.

Example 4. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}.$$

Example 5. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}.$$

11.9 Representations of Functions as Power Series

Example 1. Express $1/(1+x^2)$ as the sum of a power series and find the interval of convergence.

Example 2. Find a power series representation for $1/(x+2)$.

Example 3. Find a power series representation of $x^3/(x+2)$.

Theorem 11.9.1. *If the power series* $\sum c_n(x-a)^n$ *has radius of convergence* $R > 0$*, then the function* f *defined by*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ *and*

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii)
$$\begin{aligned}\int f(x)\,dx &= C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots \\ &= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}.\end{aligned}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R*.*

Example 4. Find the derivative of the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

Example 5. Express $1/(1-x)^2$ as a power series using differentiation. What is the radius of convergence?

Example 6. Find a power series representation for $\ln(1+x)$ and its radius of convergence.

Example 7. Find a power series representation for $f(x) = \tan^{-1} x$.

Example 8. (a) Evaluate $\int [1/(1+x^7)]dx$ as a power series.

(b) Use part (a) to approximate $\int_0^{0.5} [1/(1+x^7)]dx$ correct to within 10^{-7}.

11.10 Taylor and Maclaurin Series

Theorem 11.10.1. *If f has a power series representation (expansion) at a, that is, if*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Definition 11.10.1. The Taylor series of the function f at a (or about a or centered at a) is

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots . \end{aligned}$$

For the special case $a = 0$ the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots ,$$

which we call the Maclaurin Series.

Example 1. Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Theorem 11.10.2. *If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a, R_n is the remainder of the Taylor series, and*

$$\lim_{n \to \infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

Theorem 11.10.3 (Taylor's Inequality)**.** *If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1} \qquad \text{for } |x - a| \leq d.$$

Example 2. Prove that e^x is equal to the sum of its Maclaurin series.

Example 3. Find the Taylor series $f(x) = e^x$ at $a = 2$.

Example 4. Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

Example 5. Find the Maclaurin series for $\cos x$.

Example 6. Find the Maclaurin series for the function $f(x) = x \cos x$.

Example 7. Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\pi/3$.

Example 8. Find the Maclaurin series for $f(x) = (1 + x)^k$, where k is any real number.

Theorem 11.10.4 (The Binomial Series)**.** *If k is any real number and $|x| < 1$, then*

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

where the coefficients

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

are called the binomial coefficients.

Example 9. Find the Maclaurin series for the function $f(x) = \dfrac{1}{\sqrt{4-x}}$ and its radius of convergence.

Example 10. Find the sum of the series $\dfrac{1}{1 \cdot 2} - \dfrac{1}{2 \cdot 2^2} + \dfrac{1}{3 \cdot 2^3} - \dfrac{1}{4 \cdot 2^4} + \cdots$.

Example 11. (a) Evaluate $\int e^{-x^2}\, dx$ as an infinite series.

(b) Evaluate $\int_0^1 e^{-x^2}\, dx$ correct to within an error of 0.001.

Example 12. Evaluate $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

Example 13. Find the first three nonzero terms in the Maclaurin series for

(a) $e^x \sin x$

(b) $\tan x$

11.11 Applications of Taylor Polynomials

Example 1. (a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

(b) How accurate is this approximation when $7 \leq x \leq 9$?

Example 2. (a) What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when $-0.3 \leq x \leq 0.3$? Use this approximation to find $\sin 12^\circ$ correct to six decimal places.

(b) For what values of x is this approximation accurate to within 0.00005?

Example 3. In Einstein's theory of special relativity the mass of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of an object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0c^2.$$

(a) Show that when v is very small compared with c, this expression for K agrees with classical Newtonian physics: $K = \frac{1}{2}m_0v^2$.

(b) Use Taylor's Inequality to estimate the difference in these expressions for K when $|v| \leq 100$ m/s.

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Bibliography

[1] Stewart, James. *Calculus: Early Transcendentals.* Boston, MA, USA: Cengage Learning, 2016. Print.