Elementary Number Theory Homework #6

Replace this text with your name

Due: Replace this text with a due date

Exercise (6.1.8)**.** Show that $\sum_{d|n} 1/d = \sigma(n)/n$ for every positive integer n.

Solution: Replace this text with your solution. □

Exercise (6.1.12)**.** (a) Find the form of all positive integers n satisfying $\tau(n) = 10$. What is the smallest positive integer for which this is true?

(b) Show that there are no positive integers n satisfying $\sigma(n) = 10$. [*Hint:* Note that for $n > 1$, $\sigma(n) > n$.]

Solution: Replace this text with your solution. □

Exercise (6.2.3)**.** Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ be the prime factorization of the integer $n > 1$. If f is a multiplicative function that is not identically zero, prove that

$$\sum_{d|n} \mu(d) f(d) = (1 - f(p_1))(1 - f(p_2)) \cdots (1 - f(p_r)).$$

[*Hint:* By Theorem 6.1.4, the function F defined by $F(n) = \sum_{d|n} \mu(d) f(d)$ is multiplicative; hence, $F(n)$ is the product of the values $F(p_i^{k_i})$.]

Solution: Replace this text with your solution. □

Exercise (6.2.7)**.** The *Liouville* λ*-function* is defined by $\lambda(1) = 1$ and $\lambda(n) = (-1)^{k_1 + k_2 + \cdots + k_r}$, if the prime factorization of $n > 1$ is $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. For instance,

$$\lambda(360) = \lambda(2^3 \cdot 3^2 \cdot 5) = (-1)^{3+2+1} = (-1)^6 = 1.$$

(a) Prove that λ is a multiplicative function.

(b) Given a positive integer n, verify that

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n = m^2 \text{ for some integer } m \\ 0 & \text{otherwise.} \end{cases}$$

Solution: Replace this text with your solution. □

Exercise (6.3.1)**.** Given integers a and $b > 0$, show that there exists a unique integer r with $0 \le r < b$ satisfying $a = [a/b]b + r$.

Solution: Replace this text with your solution. □

Exercise (6.3.12)**.** Verify that the formula

$$\sum_{n=1}^{N} \lambda(n) \left[\frac{N}{n}\right] = \left[\sqrt{N}\right]$$

holds for any positive integer N.
[*Hint:* Apply Theorem 6.3.3 to the multiplicative function $F(n) = \sum_{d|n} \lambda(d)$, noting that there are $[\sqrt{n}]$ perfect squares not exceeding n.]

Solution: Replace this text with your solution. □