Elementary Number Theory Homework #7

Replace this text with your name

Due: Replace this text with a due date

Exercise (7.2.6)**.** Show that there are infinitely many integers n for which $\phi(n)$ is a perfect square.
[*Hint:* Consider the integers $n = 2^{2k+1}$ for $k = 1, 2, \ldots$.]

Solution: Replace this text with your solution. □

Exercise (7.2.13)**.** Assuming that $d \mid n$, prove that $\phi(d) \mid \phi(n)$.
[*Hint:* Work with the prime factorizations of d and n.]

Solution: Replace this text with your solution. □

Exercise (7.3.3)**.** Prove that $2^{15} - 2^3$ divides $a^{15} - a^3$ for any integer a. [*Hint:* $2^{15} - 2^3 = 5 \cdot 7 \cdot 8 \cdot 9 \cdot 13$.]

Solution: Replace this text with your solution. □

Exercise (7.3.4)**.** Show that if $\gcd(a, n) = \gcd(a - 1, n) = 1$, then

$$1 + a + a^2 + \cdots + a^{\phi(n)-1} \equiv 0 \pmod{n}.$$

[*Hint:* Recall that $a^{\phi(n)} - 1 = (a - 1)(a^{\phi(n)-1} + \cdots + a^2 + a + 1)$.]

Solution: Replace this text with your solution. □

Exercise (7.4.6)**.** Verify the formula $\sum_{d=1}^{n} \phi(d)[n/d] = n(n+1)/2$ for any positive integer n.
[*Hint:* This is a direct application of Theorems 6.3.3 and 7.4.1.]

Solution: Replace this text with your solution. □

Exercise (7.4.7)**.** If n is a square-free integer, prove that $\sum_{d|n} \sigma(d^{k-1})\phi(d) = n^k$ for all integers $k \geq 2$.

Solution: Replace this text with your solution. □