Real Analysis Homework #3

Replace this text with your name

Due: Replace this text with a due date

Exercise (3.2.11)**.** (a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) Does this result about closures extend to infinite unions of sets?

Solution: Replace this text with your solution. ☐

Exercise (3.2.13)**.** Prove that the only sets that are both open and closed are $\mathbf{R}$ and the empty set $\emptyset$.

Solution: Replace this text with your solution. ☐

Exercise (3.3.5)**.** Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

(a) The arbitrary intersection of compact sets is compact.

(b) The arbitrary union of compact sets is compact.

(c) Let A be arbitrary, and let K be compact. Then, the intersection $A \cap K$ is compact.

(d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots$ is a nested sequence of nonempty closed sets, then the intersection $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Solution: Replace this text with your solution. □

Exercise (3.3.7)**.** As some more evidence of the surprising nature of the Cantor set, follow these steps to show that the sum $C+C = \{x+y : x, y \in C\}$ is equal to the closed interval $[0, 2]$. (Keep in mind that C has zero length and contains no intervals.)

Because $C \subseteq [0, 1]$, $C + C \subseteq [0, 2]$, so we only need to prove the reverse inclusion $[0, 2] \subseteq \{x + y : x, y \in C\}$. Thus, given $s \in [0, 2]$, we must find two elements $x, y \in C$ satisfying $x + y = s$.

(a) Show that there exist $x_1, y_1 \in C_1$ for which $x_1+y_1 = s$. Show in general that, for an arbitrary $n \in \mathbf{N}$, we can always find $x_n, y_n \in C_n$ for which $x_n + y_n = s$.

(b) Keeping in mind that the sequences (x_n) and (y_n) do not necessarily converge, show how they can nevertheless be used to produce the desired x and y in C satisfying $x + y = s$.

Solution: Replace this text with your solution. □

Exercise (3.4.1)**.** If P is a perfect set and K is compact, is the intersection $P \cap K$ always compact? Always perfect?

Solution: Replace this text with your solution. $\square$

Exercise (3.4.5)**.** Let A and B be nonempty subsets of $\mathbf{R}$. Show that if there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$, then A and B are separated.

Solution: Replace this text with your solution. $\square$