Real Analysis Homework #5

Replace this text with your name

Due: Replace this text with a due date

Exercise (5.2.5)**.** Let $f_a(x) = \begin{cases} x^a & \text{if } x > 0 \\ 0 & if\ x \leq 0. \end{cases}$

(a) For which values of a is f continuous at zero?

(b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

(c) For which values of a is f twice-differentiable?

Solution: Replace this text with your solution. □

Exercise (5.2.9)**.** Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

(a) If f' exists on an interval and is not constant, then f' must take on some irrational values.

(b) If f' exists on an open interval and there is some point c where $f'(c) > 0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $f'(x) > 0$ for all $x \in V_\delta(c)$.

(c) If f is differentiable on an interval containing zero and if $\lim_{x \to 0} f'(x) = L$, then it must be that $L = f'(0)$.

Solution: Replace this text with your solution. □

Exercise (5.3.1)**.** Recall that a function $f : A \to \mathbf{R}$ is Lipschitz on A if there exists an $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M$$

for all $x \neq y$ in A.

(a) Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

(b) A contractive function f is a function defined on all of $\mathbf{R}$ where there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbf{R}$. If we add the assumption that $|f'(x)| < 1$ on $[a, b]$, does it follow that the function f from (a) is contractive on this set?

Solution: Replace this text with your solution. □

Exercise (5.3.8)**.** Assume f is continuous on an interval containing zero and differentiable for all $x \neq 0$. If $\lim_{x\to 0} f'(x) = L$, show $f'(0)$ exists and equals L.

Solution: Replace this text with your solution. □