Real Analysis Homework #6

Replace this text with your name

Due: Replace this text with a due date

Exercise (6.2.9)**.** Assume (f_n) and (g_n) are uniformly continuous sequences of functions.

(a) Show that $(f_n + g_n)$ is a uniformly continuous sequence of functions.

(b) Given an example to show that the product $(f_n g_n)$ may not converge uniformly.

(c) Prove that if there exists an $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbf{N}$, then $(f_n g_n)$ does converge uniformly.

Solution: Replace this text with your solution. □

Exercise (6.2.11)**. (Dini's Theorem).** Assume $f_n \to f$ pointwise on a compact set K and assume that for each $x \in K$ the sequence $f_n(x)$ is increasing. Follow these steps to show that if f_n and f are continuous on K, then the convergence is uniform.

(a) Set $g_n = f - f_n$ and translate the preceding hypothesis into statements about the sequence (g_n).

(b) Let $\epsilon > 0$ be arbitrary, and define $K_n = \{x \in K : g_n(x) \geq \epsilon\}$. Argue that $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$, and use this observation to finish the argument.

Solution: Replace this text with your solution. □

Exercise (6.3.1)**.** Consider the sequence of functions defined by

$$g_n(x) = \frac{x^n}{n}.$$

(a) Show (g_n) converges uniformly on $[0, 1]$ and find $g = \lim g_n$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0, 1]$.

(b) Now, show that (g'_n) converges on $[0, 1]$. Is the convergence uniform? Set $h = \lim g'_n$ and compare h and g'. Are they the same?

Solution: Replace this text with your solution. □

Exercise (6.3.3)**.** Consider the sequence of functions

$$f_n(x) = \frac{x}{1 + nx^2}.$$

(a) Find the points on $\mathbf{R}$ where each $f_n(x)$ attains its maximum and minimum value. Use this to prove (f_n) converges uniformly on $\mathbf{R}$. What is the limit function?

(b) Let $f = \lim f_n$. Compute $f'_n(x)$ and find all the values of x for which $f'(x) = \lim f'_n(x)$.

Solution: Replace this text with your solution. □

Exercise (6.4.7)**.** Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}.$$

(a) Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is continuous.

(b) Can we determine if f is twice-differentiable?

Solution: Replace this text with your solution. □

Exercise (6.4.9)**.** Let

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}$$

(a) Show that h is a continuous function defined on all of $\mathbf{R}$.

(b) Is h differentiable? If so, is the derivative function h' continuous?

Solution: Replace this text with your solution. □

Exercise (6.5.1)**.** Consider the function g defined by the power series

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots .$$

(a) Is g defined on $(-1, 1)$? Is it continuous on this set? Is g defined on $(-1, 1]$? Is it continuous on this set? What happens on $[-1, 1]$? Can the power series for $g(x)$ possibly converge for any other points $|x| > 1$? Explain.

(b) For what values of x is $g'(x)$ defined? Find a formula for g'.

Solution: Replace this text with your solution. □

Exercise (6.5.7)**.** Let $\sum a_n x^n$ be a power series with $a_n \neq 0$, and assume

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists.

(a) Show that if $L \neq 0$, then the series converges for all x in $(-1/L, 1/L)$. (The advice in Exercise 2.7.9 may be helpful.)

(b) Show that if $L = 0$, then the series converges for all $x \in \mathbf{R}$.

(c) Show that (a) and (b) continue to hold if L is replaced by the limit

$$L' = \lim_{n\to\infty} s_n \qquad \text{where} \qquad s_n = \sup \left\{ \left| \frac{a_{k+1}}{a_k} \right| : k \geq n \right\}.$$

Solution: Replace this text with your solution. □

Exercise (6.6.5)**.** (a) Generate the Taylor coefficients for the exponential function $f(x) = e^x$, and then prove that the corresponding Taylor series converges uniformly to e^x on any interval of the form $[-R, R]$.

(b) Verify the formula $f'(x) = e^x$.

(c) Use a substitution to generate the series for e^{-x}, and then informally calculate $e^x \cdot e^{-x}$ by multiplying together the two series and collecting common powers of x.

Solution: Replace this text with your solution. □

Exercise (6.6.7)**.** Find an example of each of the following or explain why no such function exists.

(a) An infinitely differentiable function $g(x)$ on all of $\mathbf{R}$ with a Taylor series that converges to $g(x)$ only for $x \in (-1, 1)$.

(b) An infinitely differentiable function $h(x)$ with the same Taylor series as $\sin(x)$ but such that $h(x) \neq \sin(x)$ for all $x \neq 0$.

(c) An infinitely differentiable function $f(x)$ on all of $\mathbf{R}$ with a Taylor series that converges to $f(x)$ if and only if $x \leq 0$.

Solution: Replace this text with your solution. □