Real Analysis Notes

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Contents

Chapter 1

The Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

Theorem 1.1.1. *There is no rational number whose square is 2.*

Proof. A rational number is any number that can be expressed in the form p/q, where p and q are integers. Assume, for contradiction, that there exist integers p and q satisfying

$$\left(\frac{p}{q}\right)^2 = 2. \tag{1}$$

We may also assume that p and q have no common factor, because, if they had one, we could simply cancel it out and rewrite the fraction in lowest terms. Equation (1) implies

$$p^2 = 2q^2. \tag{2}$$

From this, we can see that p^2 is even, and hence p must be even as well, which allows us to write $p = 2r$ where r is also an integer. Substituting $2r$ for p in equation (2) yields

$$\begin{aligned} (2r)^2 &= 2q^2 \\ 2r^2 &= q^2, \end{aligned}$$

implying that q^2 is even. However, this also implies that q is even, which contradicts the assumption that p and q have no common factor. $\square$

Example 1. (a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

a) Assume, for contradiction, there exist integers p, q s.t. $\left(\frac{p}{q}\right)^2 = 3$.

Also assume p, q have no common factor.

Then $p^2 = 3q^2$

$\Rightarrow p^2$ is a multiple of $3 \Rightarrow p$ is a multiple of 3

$\left(3 \mid (p-1)p(p+1) = p^3 - p, \text{ so } 3 \mid p^2 \Rightarrow 3 \mid p^3 \Rightarrow 3 \mid p^3 - (p^3 - p) = p\right)$

So we can write $p = 3r$ for some integer r.

$\Rightarrow (3r)^2 = 3q^2$

$3r^2 = q^2 \quad \Rightarrow q^2$ is a multiple of $3 \Rightarrow q$ is a multiple of 3

Contradiction.

In the case of $\sqrt{6}$ we get $p^2 = 6q^2 \Rightarrow p$ is a multiple of 2 and 3

$\Rightarrow q$ is a multiple of 6

Another contradiction.

b) In this case we get that p^2 is a multiple of 4, but that doesn't mean p is a multiple of 4.

Remark 1. We call the natural numbers

$$\mathbf{N} = \{1, 2, 3, 4, 5, \ldots\}.$$

The natural numbers extend to the integers

$$\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\},$$

which we extend again to the rational numbers

$$\mathbf{Q} = \left\{\text{all fractions } \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers with } q \neq 0\right\}.$$

By filling the gaps in $\mathbf{Q}$, we obtain the real numbers $\mathbf{R}$.

1.2 Some Preliminaries

Remark 1. Intuitively speaking, a set is any collection of objects. These objects are referred to as the elements of the set.
Given a set A, we write $x \in A$ if x is an element of A. If x is not an element of A, then we write $x \notin A$. Given two sets A and B, the union is written $A \cup B$ and is defined by asserting that

$$x \in A \cup B \text{ provided } x \in A \text{ or } x \in B.$$

The intersection $A \cap B$ is the set defined by the rule

$$x \in A \cap B \text{ provided } x \in A \text{ and } B.$$

Example 1. (i) There are many acceptable ways to assert the contents of a set. In the previous section, the set of natural numbers was defined by listing the elements: $\mathbf{N} = \{1, 2, 3, \ldots\}$.

(ii) Sets can also be described in words. For instance, we can define the set E to be the collection of even natural numbers.

(iii) Sometimes it is more efficient to provide a kind of rule or algorithm for determining the elements of a set. As an example, let

$$S = \{r \in \mathbf{Q} : r^2 < 2\}.$$

Read aloud, the definition of S says, "Let S be the set of all rational numbers whose squares are less than 2." It follows that $1 \in S$, $4/3 \in S$, but $3/2 \notin S$ because $9/4 \geq 2$.

Example 2. Find $\mathbf{N} \cup E$, $\mathbf{N} \cap E$, $\mathbf{N} \cap S$, and $E \cap S$.

$\mathbb{N} \cup E = \mathbb{N}$, $\mathbb{N} \cap E = E$, $\mathbb{N} \cap S = \{1\}$, $E \cap S = \emptyset$

Remark 2. The inclusion relationship $A \subseteq B$ or $B \supseteq A$ is used to indicate that every element of A is also an element of B. In this case, we say A is a subset of B or B contains A. To assert that $A = B$ means that $A \subseteq B$ and $B \subseteq A$.

Example 3. Let

$$A_1 = \mathbf{N} = \{1, 2, 3, \ldots\},$$
$$A_2 = \{2, 3, 4, \ldots\},$$
$$A_3 = \{3, 4, 5, \ldots\},$$

and, in general, for each $n \in \mathbf{N}$, define the set

$$A_n = \{n, n+1, n+2, \ldots\}.$$

The result is a nested chain of sets

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \cdots ,$$

where each successive set is a subset of all the previous ones. Notationally,

$$\bigcup_{n=1}^{\infty} A_n, \quad \bigcup_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cup A_2 \cup A_3 \cup \cdots$$

are all equivalent ways to indicate the set whose elements consist of any element that appears in at least one particular A_n. The notion of intersection has the same kind of natural extension to infinite collections of sets. What are

$$\bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \bigcap_{n=1}^{\infty} A_n$$

in this case?

$\bigcup_{n=1}^{\infty} A_n = A_1$

$\bigcap_{n=1}^{\infty} A_n = \phi$, because otherwise $m \in \bigcap_{n=1}^{\infty} A_n \Rightarrow m \in A_n \ \forall n$ but $m \notin A_{m+1}$

Remark 3. Given $A \subseteq \mathbf{R}$, the complement of A, written A^c, refers to the set of all elements of $\mathbf{R}$ not in A. Thus, for $A \subseteq \mathbf{R}$,

$$A^c = \{x \in \mathbf{R} : x \notin A\}.$$

Example 4 (De Morgan's Laws)**.** Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

a) $x \in (A \cap B)^c \Rightarrow x \notin (A \cap B) \Rightarrow x \notin A$ or $x \notin B$
$\Rightarrow x \in A^c$ or $x \in B^c$
$\Rightarrow x \in A^c \cup B^c$

b) $x \in A^c \cup B^c \Rightarrow x \in A^c$ or $x \in B^c$
$\Rightarrow x \notin A$ or $x \notin B$
$\Rightarrow x \notin (A \cap B) \Rightarrow x \in (A \cap B)^c$

c) To show $(A \cup B)^c \subseteq A^c \cap B^c$, let $x \in (A \cup B)^c \Rightarrow x \notin (A \cup B)$
$\Rightarrow x \notin A$ and $x \notin B$
$\Rightarrow x \in A^c$ and $x \in B^c$
$\Rightarrow x \in A^c \cap B^c$

To show $A^c \cap B^c \subseteq (A \cup B)^c$, let $x \in A^c \cap B^c \Rightarrow x \in A^c$ and $x \in B^c$
$\Rightarrow x \notin A$ and $x \notin B$
$\Rightarrow x \notin (A \cup B)$
$\Rightarrow x \in (A \cup B)^c$

Definition 1.2.1. Given two sets A and B, a function from A to B is a rule or mapping that takes each element $x \in A$ and associates with it a single element of B. In this case, we write $f : A \to B$. Given an element $x \in A$, the expression $f(x)$ is used to represent the element of B associated with x by f. The set A is called the domain of f. The range of f is not necessarily equal to B but refers to the subset of B given by $\{y \in B : y = f(x) \text{ for some } x \in A\}$.

Example 5. In 1829, Dirichlet proposed the unruly function

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

What are the domain and range of g?

domain of g is $\mathbb{R}$, range of g is $\mathbb{Q}$

Example 6 (Triangle Inequality)**.** The absolute value function is so important that it merits the special notation $|x|$ in place of the usual $f(x)$ or $g(x)$. It is defined for every real number via the piecewise definition

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

With respect to multiplication and division, the absolute value function satisfies

(i) $|ab| = |a||b|$ and

(ii) $|a + b| \le |a| + |b|$

for all choices of a and b. Verify these properties.

(i) if $a \geq 0$ and $b \geq 0$, then $|ab| = ab = |a||b|$

if $a \leq 0$ and $b \leq 0$, then $|ab| = ab = (-a)(-b) = |a||b|$

if $a \leq 0$ and $b \geq 0$, then $|ab| = -ab = (-a)b = |a||b|$, similar for when $a \geq 0$ and $b \leq 0$

(ii) if $a \geq 0$ and $b \geq 0$, then $a+b > 0 \Rightarrow |a+b| = a+b = |a| + |b|$

if $a \leq 0$ and $b \leq 0$, then $a+b < 0 \Rightarrow |a+b| = -(a+b) = (-a) + (-b) = |a| + |b|$

if $a < 0$, $b \geq 0$, and $a+b \geq 0$, then $|a+b| = a+b = -(-a)+b = -|a| + |b| < |a| + |b|$,

similar for when $a \geq 0$, $b < 0$, and $a+b \geq 0$

if $a < 0$, $b \geq 0$, and $a+b \leq 0$, then $|a+b| = -(a+b) = (-a) - b = |a| - |b| < |a| + |b|$,

similar for when $a \geq 0$, $b < 0$, and $a+b \leq 0$

Theorem 1.2.1. *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.*

Proof. If $a = b$, then $|a - b| = 0$, and so $|a - b| < \epsilon$ no matter what $\epsilon > 0$ is chosen.
Conversely, if $|a - b| < \epsilon$ for every $\epsilon > 0$, assume towards a contradiction that $a \neq b$. Then $|a - b| > 0$, and so we can let

$$\epsilon_0 = |a - b|.$$

However, then $|a - b| < \epsilon_0$ cannot be true, a contradiction. $\square$

Remark 4. Induction arguments are used in conjunction with the natural numbers $\mathbf{N}$ (or sometimes with the set $\mathbf{N} \cup \{0\}$). The fundamental principle behind induction is that if S is some subset of $\mathbf{N}$ with the property that

(i) S contains 1 and

(ii) whenever S contains a natural number n, it also contains $n + 1$,

then it must be that $S = \mathbf{N}$.

Example 7. Let $x_1 = 1$, and for each $n \in \mathbf{N}$ define

$$x_{n+1} = (1/2)x_n + 1.$$

Using this rule, we can compute $x_2 = (1/2)(1) + 1 = 3/2$, $x_3 = 7/4$, and it is immediately apparent how this leads to a definition of x_n for all $n \in \mathbf{N}$. The sequence just defined appears at the outset to be increasing. Use induction to prove this trend continues; that is, show

$$x_n \leq x_{n+1}$$

for all values of $n \in \mathbf{N}$.

For $n=1$, $x_1=1$ and $x_2=3/2 \Rightarrow x_1 \leq x_2 \Rightarrow 1 \in S$

If $n \in S$, then

$$x_n \leq x_{n+1}$$

$$\frac{1}{2}x_n \leq \frac{1}{2}x_{n+1}$$

$$\frac{1}{2}x_n + 1 \leq \frac{1}{2}x_{n+1} + 1$$

$$x_{n+1} \leq x_{n+2}$$

So $n+1 \in S$

1.3 The Axiom of Completeness

Axiom of Completeness. *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Definition 1.3.1. A set $A \subseteq \mathbf{R}$ is bounded above if there exists a number $b \in \mathbf{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an upper bound for A.
Similarly, the set A is bounded below if there exists a lower bound $l \in \mathbf{R}$ satisfying $l \leq a$ for every $a \in A$.

Definition 1.3.2. A real number s is the least upper bound for a set $A \subseteq \mathbf{R}$ if it meets the following two criteria:

(i) s is an upper bound for A;

(ii) if b is any upper bound for A, then $s \leq b$.

The least upper bound is also frequently called the supremum of the set A. Although the notation $s = \operatorname{lub} A$ is sometimes used, we will always write $s = \sup A$ for the least upper bound.
The greatest lower bound or infimum for A is defined in a similar way and is denote by $\inf A$.

Example 1. Let

$$A = \left\{ \frac{1}{n} : n \in \mathbf{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}.$$

Find $\sup A$ and $\inf A$.

Claim $\sup A = 1$

(i) $1 \geq \frac{1}{n} \ \forall n \in \mathbb{N}$

(ii) Suppose b is an upper bound. Then $1 \in A$, so $1 \leq b$.

$\inf A = 0$, but this is harder to prove

Definition 1.3.3. A real number a_0 is a maximum of the set A if a_0 is an element of A and $a_0 \geq A$ for all $a \in A$. Similarly, a number a_1 is a minimum of A if $a_1 \in A$ and $a_1 \leq a$ for every $a \in A$.

Example 2. Consider the open interval

$$(0,2) = \{x \in \mathbf{R} : 0 < x < 2\},$$

and the closed interval

$$[0,2] = \{x \in \mathbf{R} : 0 \leq x \leq 2\}.$$

Find the maximum, minimum, supremum, and infimum of the two intervals.

max $(0,2)$ DNE, min $(0,2)$ DNE, $\sup(0,2) = 2$, $\inf(0,2) = 0$

$\max[0,2] = 2$, $\min[0,2] = 0$, $\sup[0,2] = 2$, $\inf[0,2] = 0$

Example 3. Consider again the set

$$S = \{r \in \mathbf{Q} : r^2 < 2\},$$

Is there a least upper bound in the rational numbers? What about in the real numbers?

$b = 2, \frac{3}{2}, \frac{142}{100}, \frac{1415}{1000}, \ldots$ are all upper bounds

AoC $\Rightarrow \alpha = \sup S$

$\alpha^2 = 2$ (can't prove this yet) $\Rightarrow \alpha$ is not rational

Example 4. Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. Define the set $c + A$ by

$$c + A = \{c + a : a \in A\}.$$

Find $\sup(c + A)$.

(i) Set $s = \sup A \Rightarrow a \leq s \ \forall a \in A$

$\Rightarrow c + a \leq c + s \ \forall a \in A$

$\Rightarrow c + s$ is an upper bound for $c + A$

(ii) Let b be an upper bound for $c + A \Rightarrow c + a \leq b \ \forall a \in A$

$a \leq b - c \ \forall a \in A \Rightarrow b - c$ is an upper bound for A

$\Rightarrow s \leq b - c$

$\Rightarrow c + s \leq b$

***Lemma* 1.3.1:** Assume $s \in \mathbf{R}$ is an upper bound for a set $A \subseteq \mathbf{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.

Proof. For the forward direction, we assume $s = \sup A$ and consider $s - \epsilon$, where $\epsilon > 0$ has been arbitrarily chosen. Because $s - \epsilon < s$, part (ii) of Definition 1.3.2 implies that $s - \epsilon$ is *not* an upper bound for A. If this is the case, then there must be some element $a \in A$ for which $s - \epsilon < a$ (because otherwise $s - \epsilon$ would be an upper bound).

Conversely, assume s is an upper bound with the property that no matter how $\epsilon > 0$ is chosen, $s - \epsilon$ is no longer an upper bound for A. Notice that what this implies is that if b is any number less than s, then b is not an upper bound. (Just let $\epsilon = s - b$.) To prove that $s = \sup A$, we must verify part (ii) of Definition 1.3.2. Because we have just argued that any number smaller than s cannot be an upper bound, it follows that if b is some other upper bound for A, then $s \leq b$. $\square$

1.4 Consequences of Completeness

Theorem 1.4.1 (Nested Interval Property)**.** *For each $n \in \mathbf{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbf{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection; that is $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. In order to show that $\bigcap_{n=1}^{\infty} I_n$ is not empty, we are going to use the Axiom of Completeness (AoC) to produce a single real number x satisfying $x \in I_n$ for every $n \in \mathbf{N}$. Now, AoC is a statement about bounded sets, and the one we want to consider is the set

$$A = \{a_n : n \in \mathbf{N}\}$$

of left-hand endpoints of the intervals.

Because the intervals are nested, we see that every b_n serves as an upper bound for A. Thus, we are justified in setting

$$x = \sup A.$$

Now, consider a particular $I_n = [a_n, b_n]$. Because x is an upper bound for A, we have $a_n \leq x$. The fact that each b_n is an upper bound for A and that x is the least upper bound implies $x \leq b_n$.
Altogether then, we have $a_n \leq x \leq b_n$, which means $x \in I_n$ for every choice of $n \in \mathbf{N}$. Hence, $x \in \bigcap_{n=1}^{\infty} I_n$, and the intersection is not empty. $\square$

Example 1. Recall that $\mathbf{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbf{Q}$, then ab and $a + b$ are elements of $\mathbf{Q}$ as well.

(b) Show that if $a \in \mathbf{Q}$ and $t \in \mathbf{I}$, then $a + t \in \mathbf{I}$ and $at \in \mathbf{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is $\mathbf{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?

a) Let $a=\frac{p}{q}, b=\frac{c}{d}$ for $p,q,c,d\in\mathbb{Z}, q,d\neq 0$.

Then $ab=\frac{pc}{qd}$ for $pc,qd\in\mathbb{Z} \Rightarrow ab\in\mathbb{Q}$,

and $a+b=\frac{pd+qc}{qd}$ for $pd+qc, qd\in\mathbb{Z}\Rightarrow a+b\in\mathbb{Q}$.

b) Assume $a+t\in\mathbb{Q}$. Then $t=(t+a)-a\in\mathbb{Q}$ by (a). Contradiction.

Assume $at\in\mathbb{Q}$. Then $t=(at)(1/a)\in\mathbb{Q}$ by (a). Contradiction.

c) If $s=\sqrt{2}$ and $t=-\sqrt{2}$ then $s+t=0\in\mathbb{Q}$. If $s=\sqrt{2}$ and $t=2\sqrt{2}$ then $s+t=3\sqrt{2}\in\mathbb{I}$

If $s=\sqrt{2}$ and $t=-\sqrt{2}$ then $st=-1\in\mathbb{Q}$. If $s=\sqrt{2}$ and $t=\sqrt{3}$ then $st=\sqrt{6}\in\mathbb{I}$

$\Rightarrow \mathbb{I}$ is not closed under addition or multiplication.

Theorem 1.4.2 (Archimedean Property)**.** (i) *Given any number $x \in \mathbf{R}$, there exists an $n \in \mathbf{N}$ satisfying $n > x$.*

(ii) *Given any real number $y > 0$, there exists an $n \in \mathbf{N}$ satisfying $1/n < y$.*

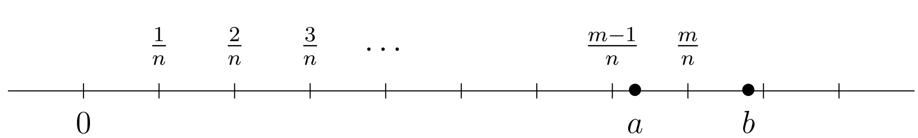
Proof. Assume, for contradiction, that $\mathbf{N}$ is bounded above. By the Axiom of Completeness (AoC), $\mathbf{N}$ should then have a least upper bound, and we can set $\alpha = \sup \mathbf{N}$. If we consider $\alpha - 1$, then we no longer have an upper bound (see Lemma 1.3.1), and therefore there exists an $n \in \mathbf{N}$ satisfying $\alpha - 1 < n$. But this is equivalent to $\alpha < n + 1$. Because $n + 1 \in \mathbf{N}$, we have a contradiction to the fact that α is supposed to be an upper bound for $\mathbf{N}$.
Part (ii) follows from (i) by letting $x = 1/y$. $\square$

Theorem 1.4.3 (Density of $\mathbf{Q}$ in $\mathbf{R}$)**.** *For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

Proof. A rational number is a quotient of integers, so we must produce $m \in \mathbf{Z}$ and $n \in \mathbf{N}$ so that

$$a < \frac{m}{n} < b. \tag{1}$$

The first step is to choose the denominator n large enough so that consecutive increments of size $1/n$ are too close together to "step over" the interval (a, b).



Using the Archimedean Property (Theorem 1.4.2), we may pick $n \in \mathbf{N}$ large enough so that

$$\frac{1}{n} < b - a. \tag{2}$$

Inequality (1) is equivalent to $na < m < nb$. With n already chosen, the idea now is to choose m to be the smallest integer greater than na. In other words, pick $m \in \mathbf{Z}$ so that

$$m - 1 \overset{(3)}{\leq} na \overset{(4)}{<} m.$$

Now, inequality (4) immediately yields $a < m/n$. Keeping in mind that inequality (2) is equivalent to $a < b - 1/n$, we can use (3) to write

$$\begin{aligned} m &\leq na + 1 \\ &< n\left(b - \frac{1}{n}\right) + 1 \\ &= nb. \end{aligned}$$

Because $m < nb$ implies $m/n < b$, we have $a < m/n < b$, as desired. □

Corollary 1.4.1. *Given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$.*

Example 2. Prove Corollary 1.4.1.

Theorem 1.4.3 $\Rightarrow \exists\, r \in \mathbb{Q}$ s.t. $a - \sqrt{2} < r < b - \sqrt{2}$

$\Rightarrow a < r + \sqrt{2} < b$

$r + \sqrt{2} \in \mathbb{I}$ by Ex 1 (b).

Theorem 1.4.4. *There exists a real number $\alpha \in \mathbf{R}$ satisfying $\alpha^2 = 2$.*

Proof. Consider the set

$$T = \{t \in \mathbf{R} : t^2 < 2\}$$

and set $\alpha = \sup T$. We are going to prove $\alpha^2 = 2$ by ruling out the possibilities $\alpha^2 < 2$ and $\alpha^2 > 2$.

Let's first assume $\alpha^2 < 2$. In search of an element of T that is larger than α, write

$$\begin{aligned}\left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha + 1}{n}.\end{aligned}$$

But now assuming $\alpha^2 < 2$ gives us a little space in which to fit the $(2\alpha+1)/n$ term and keep the total less than 2. Specifically, choose $n_0 \in \mathbf{N}$ large enough so that

$$\frac{1}{n_0} < \frac{2 - \alpha^2}{2\alpha + 1}.$$

This implies $(2\alpha + 1)/n_0 < 2 - \alpha^2$, and consequently that

$$\left(\alpha + \frac{1}{n_0}\right)^2 < \alpha^2 + (2 - \alpha^2) = 2.$$

Thus, $\alpha + 1/n_0 \in T$, contradicting the fact that α is an upper bound for T. We conclude that $\alpha^2 < 2$ cannot happen.
Now consider the case $\alpha^2 > 2$. This time, write

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}.\end{aligned}$$

Now we need to pick n_0 large enough so that

$$\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha} \quad \text{or} \quad \frac{2\alpha}{n_0} < \alpha^2 - 2.$$

With this choice of n_0, we have

$$(\alpha - 1/n_0)^2 > \alpha^2 - 2\alpha/n_0 = \alpha^2 - (\alpha^2 - 2) = 2.$$

This means $(\alpha - 1/n_0)$ is an upper bound for T. But $(\alpha - 1/n_0) < \alpha$ and $\alpha = \sup T$ is supposed to be the least upper bound. This contradiction means that the case $\alpha^2 > 2$ can be ruled out. Because we have already ruled out $\alpha^2 < 2$, we are left with $\alpha^2 = 2$ which implies $\alpha = \sqrt{2}$ exists in $\mathbf{R}$. $\square$

1.5 Cardinality

Definition 1.5.1. A function $f : A \to B$ is one-to-one (1–1) if $a_1 \neq a_2$ in A implies $f(a_1) \neq f(a_2)$ in B. The function f is onto if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Definition 1.5.2. The set A has the same cardinality as B if there exists $f : A \to B$ that is 1-1 and onto. In this case, we write $A \sim B$.

Example 1. (i) Let $E = \{2, 4, 6, \ldots\}$ be the set of even natural numbers. Show $\mathbf{N} \sim E$.

(ii) Show $\mathbf{N} \sim \mathbf{Z}$.

(i) Let $f : \mathbb{N} \to E$ be $f(n) = 2n$

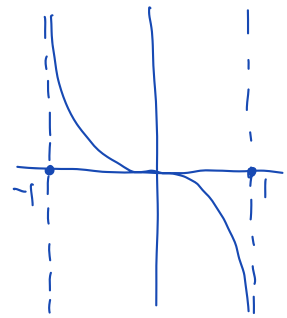
$$\begin{array}{cccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & \cdots & n & \cdots \\ & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \cdots & \Updownarrow & \\ E: & 2 & 4 & 6 & 8 & \cdots & 2n & \cdots \end{array}$$

(ii) Let $f(n) = \begin{cases} (n-1)/2 & \text{if } n \text{ is odd} \\ -n/2 & \text{if } n \text{ is even} \end{cases}$

$$\begin{array}{ccccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \\ E: & 0 & -1 & 1 & -2 & 2 & -3 & 3 & \end{array}$$

Example 2. (i) Show that $(-1,1) \sim \mathbf{R}$.

(ii) Show that $(a,b) \sim \mathbf{R}$ for any interval (a,b).

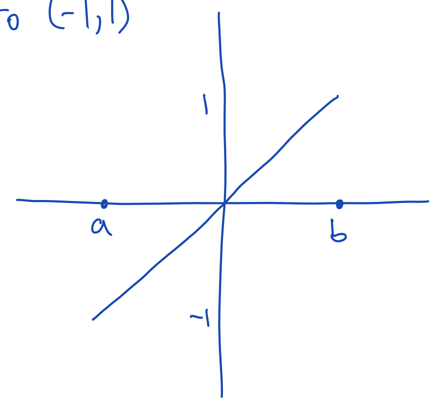


(i) $f(x) = \frac{x}{x^2-1}$ takes $(-1,1)$ onto $\mathbb{R}$ and is 1-1

(ii) a line through $(a,-1)$ to $(b,1)$ is

$f(x) = -1 + \frac{2}{b-a}(x-a)$, which takes (a,b) onto $(-1,1)$

and is 1-1



Definition 1.5.3. A set A is countable if $\mathbf{N} \sim A$. An infinite set that is not countable is called an uncountable set.

Theorem 1.5.1. (i) *The set* $\mathbf{Q}$ *is countable.*

(ii) *The set* $\mathbf{R}$ *is uncountable.*

Proof. (i) Set $A_1 = \{0\}$ and for each $n \geq 2$, let A_n be the set given by

$$A_n = \left\{ \pm \frac{p}{q} : \text{ where } p, q \in \mathbf{N} \text{ are in lowest terms with } p + q = n \right\}.$$

The first few of these sets look like

$$A_1 = \{0\}, \quad A_2 = \left\{ \frac{1}{1}, \frac{-1}{1} \right\}, \quad A_3 = \left\{ \frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1} \right\},$$

$$A_4 = \left\{ \frac{1}{3}, \frac{-1}{3}, \frac{3}{1}, \frac{-3}{1} \right\}, \quad \text{and} \quad A_5 = \left\{ \frac{1}{4}, \frac{-1}{4}, \frac{2}{3}, \frac{-2}{3}, \frac{3}{2}, \frac{-3}{2}, \frac{4}{1}, \frac{-4}{1} \right\}.$$

The crucial observation is that each A_n is *finite* and every rational number appears in exactly one of these sets. Our 1–1 correspondence with $\mathbf{N}$ is then achieved by consecutively listing the elements in each A_n.

$$\begin{array}{cccccccccccccc}
\mathbf{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbf{Q}: & \underbrace{0}_{A_1} & \multicolumn{2}{c}{\underbrace{\tfrac{1}{1} \quad -\tfrac{1}{1}}_{A_2}} & \multicolumn{4}{c}{\underbrace{\tfrac{1}{2} \quad -\tfrac{1}{2} \quad \tfrac{2}{1} \quad -\tfrac{2}{1}}_{A_3}} & \multicolumn{4}{c}{\underbrace{\tfrac{1}{3} \quad -\tfrac{1}{3} \quad \tfrac{3}{1} \quad -\tfrac{3}{1}}_{A_4}} & \tfrac{1}{4} & \cdots
\end{array}$$

We now see why every rational number appears in the correspondence exactly once. Given, say, $22/7$, we have that $22/7 \in A_{29}$. Because the set of elements in $A_1, \dots, A_{28}$ is finite, we can be confidence that $22/7$ eventually gets included in the sequence. The fact that this line of reasoning applies to any rational number p/q is our proof that the correspondence is onto. To verify that it is 1–1, we observe that the sets A_n were constructed to be disjoint so that no rational number appears twice. This completes the proof of (i).

(ii) This proof is done by contradiction. Assume that there *does* exist a 1–1, onto function $f : \mathbf{N} \to \mathbf{R}$. If we let $x_1 = f(1)$, $x_2 = f(2)$, and so on, then our assumption that f is onto means that we can write

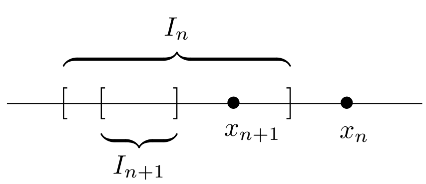
$$\mathbf{R} = \{x_1, x_2, x_3, x_4, \dots\} \tag{1}$$

and be confident that every real number appears somewhere on the list. We will now use the Nested Interval Property (Theorem 1.4.1) to produce a real number that is not there.

Let I_1 be a closed interval that *does not* contain x_1. Next, let I_2 be a closed interval, contained in I_1, which does not contain x_2. The existence of such an I_2 is easy to verify. Certainly I_1 contains two smaller *disjoint* closed intervals, and x_2 can only be in one of these. In general, given an interval I_n, construct I_{n+1} to satisfy

(i) $I_{n+1} \subseteq I_n$ and

(ii) $x_{n+1} \notin I_{n+1}$.



We now consider the intersection $\bigcap_{n=1}^{\infty} I_n$. If x_{n_0} is some real number from the list in (1), then we have $x_{n_0} \notin I_{n_0}$, and it follows that

$$x_{n_0} \notin \bigcap_{n=1}^{\infty} I_n.$$

Now, we are assuming that the list in (1) contains every real number, and this leads to the conclusion that

$$\bigcap_{n=1}^{\infty} I_n = \emptyset.$$

However, the Nested Interval Property (NIP) asserts that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. By NIP, there is at least one $x \in \bigcap_{n=1}^{\infty} I_n$ that, consequently, cannot be on the list in (1). This contradiction means that such an enumeration of $\mathbf{R}$ is impossible, and we conclude that $\mathbf{R}$ is an *uncountable* set. $\square$

Theorem 1.5.2. *If $A \subseteq B$ and B is countable, then A is either countable or finite.*

Example 3. Prove Theorem 1.5.2.

B is countable $\Rightarrow \exists f : \mathbb{N} \to B$ that is 1-1 and onto.

Let $A \subseteq B$ be infinite.

Let $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. Define $g : \mathbb{N} \to A$ inductively

$g(1) = f(n_1)$

$n_2 = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(n_1)\}\}$

$g(2) = f(n_2)$

Assume $g(k)$ is defined for $k < m$

$g(m) = f(n_m)$ where $n_m = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(n_1), \ldots f(n_{k-1})\}\}$

If $m \neq m'$ then $n_m \neq n_{m'} \Rightarrow f(n_m) = g(m) \neq g(m') = f(n_{m'})$ because f is 1-1

$\Rightarrow g : \mathbb{N} \to A$ is 1-1

Let $a \in A$. f is onto $\Rightarrow a = f(n')$ for some $n' \in \mathbb{N}$

$\Rightarrow n' \in \{n : f(n) \in A\}$

As we inductively remove the min, n' will eventually be the min

$\Rightarrow g : \mathbb{N} \to A$ is onto

$\Rightarrow A$ is countable

Theorem 1.5.3. (i) *If $A_1, A_2, \ldots, A_m$ are each countable sets, then the union $A_1 \cup A_2 \cup \cdots \cup A_m$ is countable.*

(ii) *If A_n is a countable set for each $n \in \mathbf{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable.*

Example 4. Prove Theorem 1.5.3.

(i) A_1 countable $\Rightarrow \exists\, f: \mathbb{N} \to A_1$ s.t. f is 1-1 and onto

Let $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. If $B_2 = \phi$, then $A_1 \cup A_2 = A_1$ is countable

If $B_2 = \{b_1, b_2, \ldots, b_m\}$ then define $h: \mathbb{N} \to A_1 \cup B_2$ by

$$h(n) = \begin{cases} b_n & \text{if } n \le m \\ f(n-m) & \text{if } n > m \end{cases}$$

h is 1-1 and onto because f is

If B_2 is infinite, then it is countable by Theorem 1.5.2, so $\exists\, g: \mathbb{N} \to B_2$ that is 1-1 and onto, so define $h: \mathbb{N} \to A_1 \cup B_2$ by

$$h(n) = \begin{cases} f((n+1)/2) & \text{if } n \text{ is odd} \\ g(n/2) & \text{if } n \text{ is even} \end{cases}$$

$$\begin{array}{cccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ A_1 \cup B_2 & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & \cdots \end{array}$$

h is 1-1 and onto because f and g are

Next assume the union of m countable sets is countable.

Let $A_1, A_2, \ldots, A_{m+1}$ be countable.

Then $A_1 \cup A_2 \cup \cdots \cup A_{m+1} = (A_1 \cup A_2 \cup \cdots \cup A_m) \cup A_{m+1}$ is a union of two countable sets, so it is countable.

(ii) First consider $\{A_n\}$ disjoint where $A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}$

$$\begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \cdots & & & & \\ \vdots & & & & & \end{array} \qquad \begin{array}{llllll} A_1 = a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\ A_2 = a_{21} & a_{22} & a_{23} & a_{24} & \cdots & \\ A_3 = a_{31} & a_{32} & a_{33} & \cdots & & \\ A_4 = a_{41} & a_{42} & \cdots & & & \\ A_5 = a_{51} & \cdots & & & & \\ \quad\vdots & & & & & \end{array}$$

$g: \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$ s.t. $g(n) = a_{jk}$ for n in (j,k)

If $\{A_n\}$ are not disjoint, replace A_n with $B_n = A_n \setminus \{A_1 \cup \cdots \cup A_{n-1}\}$

1.6 Cantor's Theorem

Theorem 1.6.1. *The open interval* $(0, 1) = \{x \in \mathbf{R} : 0 < x < 1\}$ *is uncountable.*

Proof. As with Theorem 1.5.1, we proceed by contradiction and assume that there does exist a function $f : \mathbf{N} \to (0, 1)$ that is 1–1 and onto. For each $m \in \mathbf{N}$, $f(m)$ is a real number between 0 and 1, and we represent it using the decimal notation

$$f(m) = .a_{m1}a_{m2}a_{m3}a_{m4}a_{m5}\ldots$$

What is meant here is that for each $m, n \in \mathbf{N}$ is the digit from the set $\{0, 1, 2, \ldots, 9\}$ that represents the nth digit in the decimal expansion of $f(m)$. The 1–1 correspondence between $\mathbf{N}$ and $(0, 1)$ can be summarized in the doubly indexed array

$$\begin{array}{cccccccccccc}
\mathbf{N} & & (0,1) & & & & & & & & \\
\hline
1 & \longleftrightarrow & f(1) & = & .\mathbf{a_{11}} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \cdots \\
2 & \longleftrightarrow & f(2) & = & .a_{21} & \mathbf{a_{22}} & a_{23} & a_{24} & a_{25} & a_{26} & \cdots \\
3 & \longleftrightarrow & f(3) & = & .a_{31} & a_{32} & \mathbf{a_{33}} & a_{34} & a_{35} & a_{36} & \cdots \\
4 & \longleftrightarrow & f(4) & = & .a_{41} & a_{42} & a_{43} & \mathbf{a_{44}} & a_{45} & a_{46} & \cdots \\
5 & \longleftrightarrow & f(5) & = & .a_{51} & a_{52} & a_{53} & a_{54} & \mathbf{a_{55}} & a_{56} & \cdots \\
6 & \longleftrightarrow & f(6) & = & .a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & \mathbf{a_{66}} & \cdots \\
\vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The key assumption about this correspondence is that *every* real number in $(0, 1)$ is assumed to appear somewhere on the list.
Now define a real number $x \in (0, 1)$ with the decimal expansion $x = .b_1b_2b_3b_4\ldots$ using the rule

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2. \end{cases}$$

To compute the digit b_1, we look at the digit a_{11} in the upper left-hand corner of the array. If $a_{11} = 2$, then we choose $b_1 = 3$; otherwise, we set $b_1 = 2$. Since a_{11} and b_1 are different, x cannot be $f(1)$. We do the same thing when computing b_2, so that a_{22} and b_2 are different, and thus x cannot be $f(2)$ either. Continuing in this fashion, the nth digit of x and $f(n)$ will always be different, so $x \neq f(n)$ for any $n \in \mathbf{N}$. But this contradicts our assumption that every real number in $(0, 1)$ appears somewhere on the list. $\square$

Example 1. Let S be the set consisting of all 0's and 1's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \ldots) : a_n = 0 \text{ or } 1\}.$$

As an example, the sequence $(1, 0, 1, 0, 1, 0, 1, 0, \ldots)$ is an element of S, as is the sequence $(1, 1, 1, 1, 1, 1, \ldots)$.

Give a rigorous argument showing that S is uncountable.

Assume $\exists\, f : \mathbb{N} \to S$ that is 1-1 and onto.

$$\begin{array}{ccccccccccc}
\mathbb{N} & & & & & & & & & & \\
\hline
1 & \leftrightarrow & f(1) & = & .a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \cdots \\
2 & \leftrightarrow & f(2) & = & .a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & \cdots \\
3 & \leftrightarrow & f(3) & = & .a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & \cdots \\
4 & \leftrightarrow & f(4) & = & .a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & \cdots \\
5 & \leftrightarrow & f(5) & = & .a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & \cdots \\
6 & \leftrightarrow & f(6) & = & .a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & \cdots \\
\vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Where $a_{mn} = 1$ or 0 for $m, n \in \mathbb{N}$.

Define $(x_n) = (x_1, x_2, x_3, \ldots) \in S$ by

$$x_n = \begin{cases} 0 & \text{if } a_{nn} = 1 \\ 1 & \text{if } a_{nn} = 0 \end{cases}$$

$f(1) \neq (x_n)$ because $a_{11} \neq x_1$

$f(2) \neq (x_n)$ because $a_{22} \neq x_2$

$f(n) \neq (x_n)$ because $a_{nn} \neq x_n \;\; \forall\, n \in \mathbb{N}$

Contradicts that f is onto.

Given a set A, the power set $P(A)$ refers to the collection of all subsets of A.

Example 2. (a) Let $A = \{a, b, c\}$. List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)

(b) If A is finite with n elements, show that $P(A)$ has 2^n elements.

a) $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

b) Let $A = \{a_1, a_2, \dots, a_n\}$

$$\left.\begin{matrix} N & N & \cdots & N \\ Y & N & \cdots & N \\ N & Y & \cdots & N \\ & \vdots & & \end{matrix}\right\} 2^n \text{ subsets}$$

Example 3. (a) Using the particular set $A = \{a, b, c\}$, exhibit two different 1–1 mappings from A into $P(A)$.

(b) Letting $C = \{1, 2, 3, 4\}$, produce an example of a 1–1 map $g : C \to P(C)$.

(c) Explain why, in parts (a) and (b), it is impossible to construct mappings that are *onto*.

a)

$a \to \{a\}$, $b \to \{a,c\}$, $c \to \{a,b,c\}$

$a \to \{b,c\}$, $b \to \emptyset$, $c \to \{a,c\}$

b)

$1 \to \{1\}$, $2 \to \{2,3,4\}$, $3 \to \{1,2,4\}$, $4 \to \{2,3\}$

c) $2^n > n$ $\forall n$, so $P(A)$ has too many elements to map onto all of them.

Theorem 1.6.2 (Cantor's Theorem)**.** *Given any set A, there does not exist a function $f : A \to P(A)$ that is onto.*

Proof. Assume, for contradiction, that $f : A \to P(A)$ is onto. Unlike the usual situation in which we have sets of numbers for the domain and range, f is a correspondence between a set and its power set. For each element $a \in A$, $f(a)$ is a particular *subset* of A. The assumption that f is onto means that every subset of A appears as $f(a)$ for some $a \in A$. To arrive at a contradiction, we will produce a subset $B \subseteq A$ that is not equal to $f(a)$ for any $a \in A$.
Construct B using the following rule. For each element $a \in A$, consider the subset $f(a)$. This subset of A may contain the element a or it may not. This

depends on the function f. If $f(a)$ does not contain a, then we include a in our set B. More precisely, let

$$B = \{a \in A : a \notin f(a)\}.$$

Because we have assumed that our function $f : A \to P(A)$ is onto, it must be that $B = f(a')$ for some $a' \in A$. The contradiction arises when we consider whether or not a' is an element of B. If $a' \in B$, then $a' \in f(a')$ since $B = f(a')$. However, by the definition of B, we have $a' \notin f(a')$, a contradiction. On the other hand, if $a' \notin B$, then $a' \notin f(a')$, and again we have a contradiction by the definition of B because this implies that $a' \in B$. □

Example 4. Answer each of the following by establishing a 1–1 correspondence with a set of known cardinality.

(a) Is the set of all functions from $\{0, 1\}$ to $\mathbf{N}$ countable or uncountable?

(b) Is the set of all functions from N to $\{0, 1\}$ countable or uncountable?

(c) Given a set B, a subset $\mathcal{A}$ of $P(B)$ is called an <u>antichain</u> if no element of $\mathcal{A}$ is a subset of any other element of $\mathcal{A}$. Does $P(\mathbf{N})$ contain an uncountable antichain?

a) $f \in A \Rightarrow f : \{0,1\} \to \mathbb{N}$

$\Updownarrow$

$\{(m,n) : m,n \in \mathbb{N}\}$ by taking $m = f(0)$ and $n = f(1)$, this is 1-1 and onto

$\{(m,n) : m,n \in \mathbb{N}\} = \bigcup_{n=1}^{\infty} \{(m,n) : m \in \mathbb{N}\} \Rightarrow \{(m,n) : m,n \in \mathbb{N}\}$ is countable

b) $f : \mathbb{N} \to \{0,1\}$ is an infinite sequence of 0's and 1's, e.g., $\{0,1,1,0,1,0,0,0,1,\dots\}$, so uncountable

c) Let $E = \{2,4,\dots,2n,\dots\}$ and $O = \{1,3,\dots,2n-1,\dots\}$

Let S be the set of infinite sequences of 0's and 1's

For each $s \in S$ where $s = (s_1, s_2, s_3, \dots)$ construct $A_s \subseteq \mathbb{N}$ so that

$2n \in A_s$ iff $s_n = 1$ and

$2n-1 \in A_s$ iff $s_n = 0$

$E \cap O = \phi$ and $\mathbb{N} = E \cup O \Rightarrow \{A_s : s \in S\}$ is an uncountable antichain

Chapter 2

Sequences and Series

2.1 Discussion: Rearrangements of Infinite Series

$$.69 \approx S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots \qquad s_1 = 1,\ s_2 = \frac{1}{2},\ s_3 = \frac{5}{6},\ s_4 = \frac{7}{12}, \ldots$$

$$+\ \frac{1}{2}S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots$$

$$\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} \cdots \approx 1.03$$

$$\sum_{n=0}^{\infty} (-1/2)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \frac{1}{256} \cdots = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}$$

$$1 + \frac{1}{4} - \frac{1}{2} + \frac{1}{16} + \frac{1}{64} - \frac{1}{8} + \frac{1}{256} + \frac{1}{1024} - \frac{1}{32} \cdots = \frac{2}{3}$$

$\{a_{ij}\} = 1/2^{j-i}$ if $j > i$, $a_{ij} = -1$ if $j = i$ and $a_{ij} = 0$ if $j < i$

$$\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & \cdots \\ 0 & 0 & 0 & -1 & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{i=1}^{\infty} (0) = 0$$

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{\infty}\left(-\frac{1}{2^{j-1}}\right) = -2$$

$$\sum_{n=1}^{\infty} (-1)^n \qquad (-1+1) + (-1+1) + (-1+1) + \cdots = 0 + 0 + 0 + \cdots = 0 \qquad -1 + (1-1) + (1-1) + (1+1) + \cdots = -1 + 0 + 0 + 0 + \cdots = -1$$

2.2 The Limit of a Sequence

Definition 2.2.1. A sequence is a function whose domain is $\mathbf{N}$.

Example 1. Each of the following are common ways to describe a sequence.

(i) $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$,

(ii) $(\frac{1+n}{n})_{n=1}^{\infty} = (\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \cdots)$,

(iii) (a_n), where $a_n = 2^n$ for each $n \in \mathbf{N}$,

(iv) (x_n), where $x_1 = 2$ and $x_{n+1} = \frac{x_n+1}{2}$.

Definition 2.2.2 (Convergence of a Sequence)**.** A sequence (a_n) converges to a real number a if, for every positive number ϵ, there exists an $N \in \mathbf{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

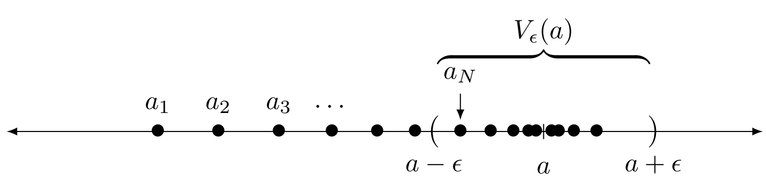
Remark 1. To indicate that (a_n) converges to a, we usually write either $\lim a_n = a$ or $(a_n) \to a$. The notation $\lim_{n\to\infty} a_n = a$ is also standard.

Definition 2.2.3. Given a real number $a \in \mathbf{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbf{R} : |x - a| < \epsilon\}$$

is called the ϵ-neighborhood of a.

Definition 2.2.4 (Convergence of a Sequence: Topological Version)**.** A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood contains all but a finite number of the terms of (a_n).



Example 2. Consider the sequence (a_n), where $a_n = 1/\sqrt{n}$. Prove that

$$\lim\left(\frac{1}{\sqrt{n}}\right) = 0.$$

Say $\varepsilon = \frac{1}{10}$. Then $a_{100} = \frac{1}{10} \Rightarrow$ if $n > 100$, then $a_n \in \left(-\frac{1}{10}, \frac{1}{10}\right)$.

If $\varepsilon = \frac{1}{50}$ then we need $\frac{1}{\sqrt{n}} < \frac{1}{50}$, which happens as long as $n > 50^2 = 2500$.

In general, we want $\frac{1}{\sqrt{n}} < \varepsilon$, which is the same as $n > \frac{1}{\varepsilon^2}$.

Let $\varepsilon > 0$ and choose N s.t. $N > \frac{1}{\varepsilon^2}$. Let $n \geq N$. Then,

$$n > \frac{1}{\varepsilon^2} \Rightarrow \frac{1}{\sqrt{n}} < \varepsilon \Rightarrow |a_n - 0| < \varepsilon$$

Example 3. Show

$$\lim\left(\frac{n+1}{n}\right) = 1.$$

We need $\left|\frac{n+1}{n} - 1\right| < \varepsilon$.

Since $\left|\frac{n+1}{n} - 1\right| = \frac{1}{n}$, this is the same as $\frac{1}{n} < \varepsilon$ or $n > \frac{1}{\varepsilon}$.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$.

Let $n \in \mathbb{N}$ satisfy $n \geq N$.

$$n \geq N \Rightarrow n > \frac{1}{\varepsilon}$$
$$\Rightarrow \left|\frac{n+1}{n} - 1\right| < \varepsilon$$

Theorem 2.2.1 (Uniqueness of Limits)**.** *The limit of a sequence, when it exists, must be unique.*

Example 4. Prove Theorem 2.2.1.

Let $a_n \to a$ and $a_n \to b$. Suppose $a \neq b$, so $|a-b|>0$.
Then for $\varepsilon = \frac{|a-b|}{2}$ $\exists\, N_1 \in \mathbb{N}$ s.t. $\forall\, n \geq N_1$: $|a_n - a| < \varepsilon$.
Similarly, $\exists\, N_2 \in \mathbb{N}$ s.t. $\forall\, n \geq N_2$: $|a_n - b| < \varepsilon$.
Choose $N = \max\{N_1, N_2\}$. Then for $n \geq N$, $|a_n - a| < \varepsilon$ and $|a_n - b| < \varepsilon$.
By the triangle inequality,

$$|a-b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| < 2\varepsilon = |a-b|$$

Contradiction.

Example 5. Show that the sequence

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \cdots\right)$$

does not converge to 0.

If $\varepsilon = \frac{1}{2}$, then after $N=3$, $a_n \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.
Similar for $\varepsilon = \frac{1}{4}$.
If $\varepsilon = \frac{1}{10}$, then there is no $N \in \mathbb{N}$ for which $a_n \in \left(-\frac{1}{10}, \frac{1}{10}\right)$.
$\Rightarrow a_n \not\to 0$.
If $\varepsilon = \frac{1}{10}$, then $|a_n - \frac{1}{5}| < \varepsilon$ produces $\left(\frac{1}{10}, \frac{3}{10}\right)$, so there is also no $N \in \mathbb{N}$ for which $a_n \to \frac{1}{5}$.

Definition 2.2.5. A sequence that does not converge is said to <u>diverge</u>.

2.3 The Algebraic and Order Limit Theorems

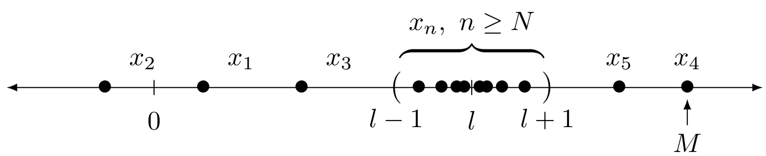
Definition 2.3.1. A sequence (x_n) is <u>bounded</u> if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbf{N}$.

Theorem 2.3.1. *Every convergent sequence is bounded.*

Proof. Assume (x_n) converges to a limit l. This means that given a particular value of ϵ, say $\epsilon = 1$, we know there must exist an $N \in \mathbf{N}$ such that if $n \geq N$, then x_n is in the interval $(l-1, l+1)$. Not knowing whether l is positive or negative, we can certainly conclude that

$$|x_n| < |l| + 1$$

for all $n \geq N$.



Because there are only a finite number of terms before the Nth term, we let

$$M = \max\{|x_1|, |x_2|, |x_3|, \ldots, |x_{N-1}|, |l| + 1\}.$$

It follows that $|x_n| \leq M$ for all $n \in \mathbf{N}$, as desired. □

Theorem 2.3.2 (Algebraic Limit Theorem)**.** *Let* $\lim a_n = a$, *and* $\lim b_n = b$. *Then,*

(i) $\lim(ca_n) = ca$, *for all* $c \in \mathbf{R}$*;*

(ii) $\lim(a_n + b_n) = a + b$*;*

(iii) $\lim(a_n b_n) = ab$*;*

(iv) $\lim(a_n/b_n) = a/b$, *provided* $b \neq 0$.

Proof. (i) Consider the case where $c \neq 0$. First, we let ϵ be some arbitrary positive number. Our goal is to find some point in the sequence (ca_n) after which we have

$$|ca_n - ca| < \epsilon.$$

Now,

$$|ca_n - ca| = |c||a_n - a|.$$

We are given that $(a_n) \to a$, we know we can make $|a_n - a|$ as small we like. In particular, we can choose an N such that

$$|a_n - a| < \frac{\epsilon}{|c|}$$

whenever $n \geq N$. To see that this N indeed works, observe that, for all $n \geq N$,

$$|ca_n - ca| = |c||a_n - a| < |c|\frac{\epsilon}{|c|} = \epsilon.$$

The case $c = 0$ reduces to showing that the constant sequence $(0, 0, 0, \ldots)$ converges to 0, which is easily verified.
(ii) To prove this statement, we need to argue that the quantity

$$|(a_n + b_n) - (a + b)|$$

can be made less than an arbitrary ϵ using the assumptions that $|a_n - a|$ and $|b_n - b|$ can be made as small as we like for large n. The first step is to use the triangle inequality (Example 6) to say

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|.$$

Again, we let $\epsilon > 0$ be arbitrary. Using the hypothesis that $(a_n) \to a$, we know there exists an N_1 such that

$$|a_n - a| < \frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N_1.$$

Likewise, the assumption that $(b_n) \to b$ means that we can choose an N_2 so that

$$|b_n - b| < \frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N_2.$$

By choosing $N = \max\{N_1, N_2\}$, we ensure that if $n \geq N$, then $n \geq N_1$ and $n \geq N_2$. This allows us to conclude that

$$\begin{aligned} |(a_n + b_n) - (a + b)| &\leq |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $n \geq N$, as desired.
(iii) To show that $(a_n b_n) \to ab$, we begin by observing that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b|. \end{aligned}$$

Let $\epsilon > 0$ be arbitrary. For the piece on the right-hand side ($|a||b_n - b|$), if $a \neq 0$ we can choose N_1 so that

$$n \geq N_1 \quad \text{implies} \quad |b_n - b| < \frac{1}{|a|}\frac{\epsilon}{2}.$$

(The case when $a = 0$ is handled in Example 1.) Using the fact that convergence sequences are bounded, we know there exists a bound $M > 0$ satisfying $|b_n| \leq M$ for all $n \in \mathbf{N}$. Now, we can choose N_2 so that

$$|a_n - a| < \frac{1}{M}\frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N_2.$$

To finish the argument, pick $N = \max\{N_1, N_2\}$, and observe that if $n \geq N$, then

$$\begin{aligned}
|a_n b_n - ab| &\leq |a_n b_n - ab_n| + |ab_n - ab| \\
&= |b_n||a_n - a| + |a||b_n - b| \\
&\leq M|a_n - a| + |a||b_n - b| \\
&< M\left(\frac{\epsilon}{M2}\right) + |a|\left(\frac{\epsilon}{|a|2}\right) = \epsilon.
\end{aligned}$$

(iv) This final statement will follow from (iii) if we can prove that

$$(b_n) \to b \quad \text{implies} \quad \left(\frac{1}{b_n}\right) \to \frac{1}{b}$$

whenever $b \neq 0$. We begin by observing that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b||b_n|}.$$

Consider the particular value $\epsilon_0 = |b|/2$. Because $(b_n) \to b$, there exists an N_1 such that $|b_n - b| < |b|/2$ for all $n \geq N_1$. This implies $|b_n| > |b|/2$.
Next, choose N_2 so that $n \geq N_2$ implies

$$|b_n - b| < \frac{\epsilon|b|^2}{2}.$$

Finally, if we let $N = \max\{N_1, N_2\}$, then $n \geq N$ implies

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = |b - b_n|\frac{1}{|b||b_n|} < \frac{\epsilon|b|^2}{2}\frac{1}{|b|\frac{|b|}{2}} = \epsilon. \qquad \square$$

Example 1.

(a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

(c) Use (a) to prove Theorem 2.3.2, part (iii), for the case when $a = 0$.

a) (a_n) bounded $\Rightarrow \exists K$ s.t. $|a_n| \leq K$.

Let $\varepsilon > 0$. We need N s.t. when $n \geq N$, $|a_n b_n - 0| < \varepsilon$.

$$|a_n b_n - 0| = |a_n||b_n| \leq K|b_n|$$

$(b_n) \to 0 \Rightarrow \exists N$ s.t.

$$|b_n| < \frac{\varepsilon}{K}.$$

With this N, $|a_n b_n - 0| \leq K|b_n| < K\frac{\varepsilon}{K} = \varepsilon$ for all $n \geq N$.

$\Rightarrow (a_n b_n) \to 0$.

We cannot use the Algebraic Limit Theorem because that requires both (a_n) and (b_n) be convergent.

b) No, if $(a_n) = (1, -1, 1, -1, \dots)$ then $(a_n b_n)$ will not converge.

c) Convergent sequences are bounded, so if $(a_n) \to 0$ and $(b_n) \to b$, then $(a_n b_n) \to 0$ by part (a).

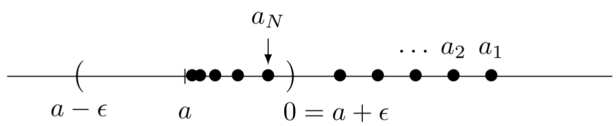
Theorem 2.3.3 (Order Limit Theorem)**.** *Assume* $\lim a_n = a$ *and* $\lim b_n = b$.

(i) *If* $a_n \geq 0$ *for all* $n \in \mathbf{N}$, *then* $a \geq 0$.

(ii) *If* $a_n \leq b_n$ *for all* $n \in \mathbf{N}$, *then* $a \leq b$.

(iii) *If there exists* $c \in \mathbf{R}$ *for which* $c \leq b_n$ *for all* $n \in \mathbf{N}$, *then* $c \leq b$. *Similarly, if* $a_n \leq c$ *for all* $n \in \mathbf{N}$, *then* $a \leq c$.

Proof. We will prove this by contradiction; thus, let's assume $a < 0$. Consider the particular value $\epsilon = |a|$. The definition of convergence guarantees that we can find an N such that $|a_n - a| < |a|$ for all $n \geq N$. In particular, this would mean that $|a_N - a| < |a|$, which implies $a_N < 0$. This contradicts our hypothesis that $a_N \geq 0$. We therefore conclude that $a \geq 0$.



(ii) The Algebraic Limit Theorem ensures that the sequence $(b_n - a_n)$ converges to $b - a$. Because $b_n - a_n \geq 0$, we can apply part (i) to get $g - a \geq 0$.
(iii) Take $a_n = c$ (or $b_n = c$) for all $n \in \mathbf{N}$, and apply (ii). ☐

Example 2. Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

($\Rightarrow$) Let $\varepsilon > 0$ and let $(z_n) \to L$.

We need N s.t. $n \geq N \Rightarrow |y_n - L| < \varepsilon$.

$(z_n) \to L \Rightarrow \exists N$ s.t. $n \geq N \Rightarrow |z_n - L| < \varepsilon$.

$y_n = z_{2n} \Rightarrow \exists N$ s.t. $n \geq N \Rightarrow |y_n - L| < \varepsilon$.

The argument is similar for (x_n).

($\Leftarrow$) Let $\varepsilon > 0$ and let $(x_n) \to L$ and $(y_n) \to L$.

We need N s.t. $n \geq N \Rightarrow |z_n - L| < \varepsilon$

$(x_n) \to L \Rightarrow \exists N_1$ s.t. $n \geq N_1 \Rightarrow |x_n - L| < \varepsilon$.

$(y_n) \to L \Rightarrow \exists N_2$ s.t. $n \geq N_2 \Rightarrow |y_n - L| < \varepsilon$.

Let $N = \max\{2N_1, 2N_2\}$.

Then $n \geq N \Rightarrow |z_n - L| < \varepsilon$.

2.4 The Monotone Convergence Theorem and Infinite Series

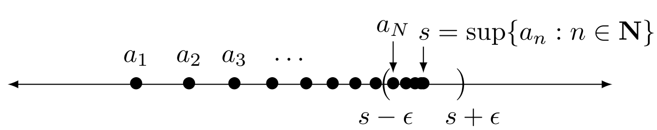
Definition 2.4.1. A sequence (a_n) is increasing if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$ and decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$. A sequence is monotone if it is either increasing or decreasing.

Theorem 2.4.1 (Monotone Convergence Theorem)**.** *If a sequence is monotone and bounded, then it converges.*

Proof. Let (a_n) be monotone and bounded. Let's assume the sequence is increasing (the decreasing case is handled similarly), and consider the *set* of points $\{a_n : n \in \mathbf{N}\}$. By assumption, this set is bounded, so we can let

$$s = \sup\{a_n : n \in \mathbf{N}\}.$$

It seems reasonable to claim that $\lim a_n = s$.



To prove this, let $\epsilon > 0$. Because s is the least upper bound for $\{a_n : n \in \mathbf{N}\}$, $s - \epsilon$ is not an upper bound, so there exists a point in the sequence a_N such that $s - \epsilon < a_N$. Now, the fact that (a_n) is increasing implies that if $n \geq N$, then $a_N \leq a_n$. Hence,

$$s - \epsilon < a_N \leq a_n \leq s < s + \epsilon,$$

which implies $|a_n - s| < \epsilon$, as desired. □

Definition 2.4.2 (Convergence of a Series)**.** Let (b_n) be a sequence. An infinite series is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots .$$

We define the corresponding sequence of partial sums (s_m) by

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m,$$

and say that the series $\sum_{n=1}^{\infty} b_n$ converges to B if the sequence (s_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

Example 1. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

converges.

$$S_m = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{m^2}$$
$$= 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \cdots + \frac{1}{m^2}$$
$$< 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3} + \cdots + \frac{1}{m(m-1)}$$
$$= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{(m-1)} - \frac{1}{m}\right)$$
$$= 1 + 1 - \frac{1}{m}$$
$$< 2$$

$\Rightarrow$ 2 is an upper bound for (S_m)

$\Rightarrow$ (S_m) converges by the Monotone Convergence Theorem

Example 2 (Harmonic Series)**.** Show that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

$$S_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$
$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 2$$
$$S_8 > 2\frac{1}{2}$$
$$S_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right)$$
$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^k} + \cdots + \frac{1}{2^k}\right)$$
$$= 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \cdots + 2^{k-1}\left(\frac{1}{2^k}\right)$$
$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}$$
$$= 1 + k\left(\frac{1}{2}\right), \text{ which is unbounded}$$

Theorem 2.4.2 (Cauchy Condensation Test)**.** *Suppose* (b_n) *is decreasing and satisfies* $b_n \geq 0$ *for all* $n \in \mathbf{N}$*. Then, the series* $\sum_{n=1}^{\infty} b_n$ *converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_16 + \cdots$$

converges.

Proof. First, assume that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges. Theorem 2.3.1 guarantees that the partial sums

$$t_k = b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k}$$

are bounded; that is, there exists an $M > 0$ such that $t_k \leq M$ for all $k \in \mathbf{N}$. We want to prove that $\sum_{n=1}^{\infty} b_n$ converges. Because $b_n \geq 0$, we know that the partial sums are increasing, so we only need to show that

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m$$

is bounded.

Fix m and let k be large enough to ensure $m \leq 2^{k+1} - 1$. Then, $s_m \leq s_{2^{k+1}-1}$ and

$$\begin{aligned} s_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \cdots + (b_{2^k} + \cdots + b_{2^{k+1}-1}) \\ &\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + \cdots + (b_{2^k} + \cdots + b_{2^k}) \\ &= b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k} = t_k. \end{aligned}$$

Thus, $s_m \leq t_k \leq M$, and the sequence (s_m) is bounded. By the Monotone Convergence Theorem, we can conclude that $\sum_{n=1}^{\infty} b_n$ converges.

We will show that if $\sum_{n=0}^{\infty} 2^n b_{2n}$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges by again exploiting a relationship between the partial sums

$$s_m = b_1 + b_2 + \cdots + b_m, \quad \text{and} \quad t_k = b_1 + 2b_2 + \cdots + 2^k b_{2^k}.$$

Because $\sum_{n=0}^{\infty} 2^n b_{2n}$ diverges, its monotone sequence of partial sums (t_k) must be unbounded. To show that (s_m) is unbounded it is enough to show that for all $k \in \mathbf{N}$, there is a term s_m satisfying $s_m \geq t_k/2$. This argument is similar to the one for the forward direction, only to get the inequality to go the other way we group the terms in s_m so that the *last* (and hence smallest) term in each group is of the form b_{2^k}.

Given an arbitrary k, we focus our attention on s_{2^k} and observe that

$$\begin{aligned} s_{2^k} &= b_1 + b_2 + (b_3 + b_4) + (b_5 + b_6 + b_7 + b_8) + \cdots + (b_{2^{k-1}+1} + \cdots + b_{2^k}) \\ &\geq b_1 + b_2 + (b_4 + b_4) + (b_8 + b_8 + b_8 + b_8) + \cdots + (b_{2^k} + \cdots + b_{2^k}) \\ &= b_1 + b_2 + 2b_4 + 4b_8 + \cdots + 2^{k-1} b_{2^k} \\ &= \frac{1}{2}(2b_1 + 2b_2 + 4b_4 + 8b_8 + \cdots + 2^k b_{2^k}) \\ &= b_1/2 + t_k/2. \end{aligned}$$

Because (t_k) is unbounded, the sequence (s_m) must also be unbounded and cannot converge. Therefore, $\sum_{n=1}^{\infty} b_n$ diverges.c $\square$

Corollary 2.4.1. *The series* $\sum_{n=1}^{\infty} 1/n^p$ *converges if and only if* $p > 1$.

2.5 Subsequences and the Bolzano–Weierstrass Theorem

Definition 2.5.1. Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < n_5 < \ldots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \ldots)$$

is called a <u>subsequence</u> of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbf{N}$ indexes the subsequence.

Theorem 2.5.1. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Proof. Assume $(a_n) \to a$, and let (a_{n_k}) be a subsequence. Given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ whenever $n \geq N$. Because $n_k \geq k$ for all k, the same N will suffice for the subsequence; that is, $|a_{n_k} - a| < \epsilon$ whenever $k \geq N$. $\square$

Example 1. Show $\lim(b^n) = 0$ if and only if $-1 < b < 1$.

If $b=0$, $(b^n)=(0,0,0,\ldots)$, so let $0<b<1$.

$b>b^2>b^3>b^4>\cdots>0 \Rightarrow (b^n)$ is decreasing and bounded below

$\Rightarrow (b^n)\to \ell$ satisfying $b>\ell\geq 0$ by MCT

$\Rightarrow (b^{2n})\to\ell$

$b^{2n}=b^n\cdot b^n \Rightarrow (b^{2n})\to \ell\cdot\ell=\ell^2$

$\Rightarrow \ell^2=\ell$

$\Rightarrow \ell=0$

Next let $-1<b<0$ and let $\varepsilon>0$.

Set $a=|b|$. Since $(a_n)\to 0$, we can choose N s.t. $n\geq N \Rightarrow |a^n-0|<\varepsilon$.

Then $|b^n-0|=|b^n|=|a^n|<\varepsilon$ when $n\geq N$.

Example 2 (Divergence Criterion)**.** Use Theorem 2.5.1 to show that the sequence

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \cdots\right)$$

diverges.

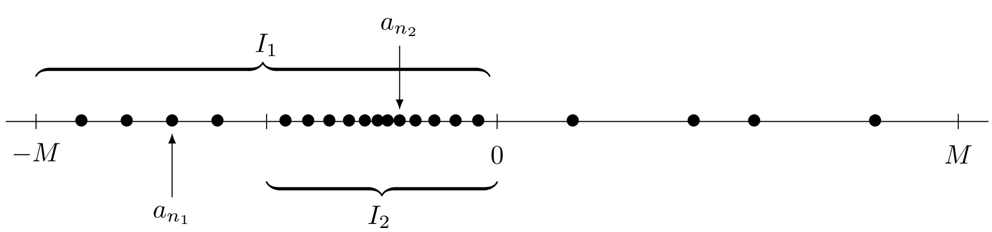
$$\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \dots\right) \to \frac{1}{5}$$

$$\left(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \dots\right) \to -\frac{1}{5}$$

There are two subsequences that converge to different limits, so the original sequence diverges.

Theorem 2.5.2 (Bolzano–Weierstrass Theorem)**.** *Every bounded sequence contains a convergent subsequence.*

Proof. Let (a_n) be a bounded sequence so that there exists $M > 0$ satisfying $|a_n| \leq M$ for all $n \in \mathbf{N}$. Bisect the closed interval $[-M, M]$ into the two closed intervals $[-M, 0]$ and $[0, M]$. Now, it must be that at least one of these closed intervals contains an infinite number of the terms in the sequence (a_n). Select a half for which this is the case and label that interval as I_1. Then, let a_{n_1} be some term in the sequence (a_n) satisfying $a_{n_1} \in I_1$.



Next, we bisect I_1 into closed intervals of equal length, and let I_2 be a half that again contains an infinite number of terms of the original sequence. Because there are an infinite number of terms from (a_n) to choose from, we can select an a_{n_2} from the original sequence with $n_2 > n_1$ and $a_{n_2} \in I_2$. In general, we construct the closed interval I_k by taking a half of I_{k-1} containing an infinite number of terms of (a_n) and then select $n_k > n_{k-1} > \cdots n_2 > n_1$ so that $a_{n_k} \in I_k$.

We want to argue that (a_{n_k}) is a convergent subsequence, but we need a candidate for the limit. The sets

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

form a nested sequence of closed intervals, and by the Nested Interval Property there exists at least one point $x \in \mathbf{R}$ contained in every I_k. This provides us with the candidate we were looking for. It just remains to show that $(a_{n_k}) \to x$. Let $\epsilon > 0$. By construction, the length of I_k is $M(1/2)^{k-1}$ which converges to zero. (This follows from Example 1 and the Algebraic Limit Theorem.) Choose N so that $k \geq N$ implies that the length of I_k is less than ϵ. Because x and a_{n_k} are both in I_k, it follows that $|a_{n_K} - x| < \epsilon$. □

Example 3. Assume the Nested Interval Property is true and use it to provide a proof of the Axiom of Completeness. To prevent the argument from being circular, assume also that $(1/2^n) \to 0$. (Why precisely is this last assumption needed to avoid circularity?)

Let A be a bounded set and b_1 an upper bound on A.

If $b_1 \in A$, then $b_1 = \sup A$.

If $b_1 \notin A$, then $\exists\, a_1 \in A$ s.t. $a_1 < b_1$. Let $I_1 = [a_1, b_1]$.

If the midpoint $m_1 = \frac{a_1 + b_1}{2}$ is an upper bound, let $I_2 = [a_2, b_2] = [a_1, m_1]$

Otherwise, let $I_2 = [a_2, b_2] = [m_1, b_1]$

Continue this process, so that if $m_n = \frac{a_n + b_n}{2}$ is an upper bound,

let $I_{n+1} = [a_{n+1}, b_{n+1}] = [a_n, m_n]$, and otherwise let $I_{n+1} = [a_{n+1}, b_{n+1}] = [m_n, b_n]$

This yields infinitely many intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ with lengths proportional to $(1/2)^n$.

By the NIP, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$, and since $(1/2^n) \to 0$ there is a single $b \in \bigcap_{n=1}^{\infty} I_n$.

Then $b \geq a_n\ \forall n \Rightarrow b$ is an upper bound,

and $b \leq b_n\ \forall n \Rightarrow b = \sup A$.

The assumption that $(1/2^n) \to 0$ is needed to avoid using the Archimedean Property to find b, since it was proved using AoC.

2.6 The Cauchy Criterion

Definition 2.6.1. A sequence (a_n) is called a Cauchy sequence if, for every $\epsilon > 0$, there exists an $n \in \mathbf{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \epsilon$.

Theorem 2.6.1. *Every convergent sequence is a Cauchy sequence.*

Proof. Assume (x_n) converges to x, and let $\epsilon > 0$ be arbitrary. Because $(x_n) \to x$, there exists $N \in \mathbf{N}$ such that $n, m \geq N$ implies $|x_n - x| < \epsilon/2$ and $x_m - x| < \epsilon/2$. By the triangle inequality,

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &\leq |x_n - x| + |x_m - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, $|x_n - x_m| < \epsilon$ whenever $n, m \geq N$, and (x_n) is a Cauchy sequence. □

***Lemma* 2.6.1:** Cauchy sequences are bounded.

Proof. Given $\epsilon = 1$, there exists an N such that $|x_m - x_n| < 1$ for all $m, n \geq N$. Thus, we must have $|x_n| < |x_N| + 1$ for all $n \geq N$. It follows that

$$M = \max\{|x_1|, |x_2|, |x_3|, \ldots, |x_{N-1}|, |x_N| + 1\}$$

is a bound for the sequence (x_n). □

Theorem 2.6.2 (Cauchy Criterion). *A sequence converges if and only if it is a Cauchy sequence.*

Proof. $(\Rightarrow)$ This direction is Theorem 2.6.1.
$(\Leftarrow)$ For this direction, we start with a Cauchy sequence (x_n). Lemma 2.6.1 guarantees that (x_n) is bounded, so we may use the Bolzano-Weierstrass Theorem to produce a convergent subsequence (x_{n_k}). Set

$$x = \lim x_{n_k}.$$

Let $\epsilon > 0$. Because (x_n) is Cauchy, there exists N such that

$$|x_n - x_m| < \frac{\epsilon}{2}$$

whenever $m, n \geq N$. Now, we also know that $(x_{n_k}) \to x$, so choose a term in this subsequence, call it x_{n_K}, with $n_K \geq N$ and

$$|x_{n_K} - x| < \frac{\epsilon}{2}.$$

To see that N has the desired property (for the original sequence (x_n)), observe that if $n \geq N$, then

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_K} + x_{x_K} - x| \\ &\leq |x_n - x_{n_K}| + |x_{n_K} - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

Example 1. If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.2, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

(b) Do the same for the product $(x_n y_n)$.

a) Let $\varepsilon > 0$. We need to find N s.t. $n, m \geq N \Rightarrow |(x_n + y_n) - (x_m + y_m)| < \varepsilon$.

$$(x_n) \ \& \ (y_n) \text{ Cauchy} \Rightarrow \begin{cases} \exists N_1 \text{ s.t. } n, m \geq N_1 \Rightarrow |x_n - x_m| < \frac{\varepsilon}{2} \\ \exists N_2 \text{ s.t. } n, m \geq N_2 \Rightarrow |y_n - y_m| < \frac{\varepsilon}{2} \end{cases}$$

Set $N = \max\{N_1, N_2\}$.

Then $n, m \geq N \Rightarrow |(x_n + y_n) - (x_m + y_m)| \leq |x_n - x_m| + |y_n - y_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

b) Let $\varepsilon > 0$. We need to find N s.t. $n, m \geq N \Rightarrow |x_n y_n - x_m y_m| < \varepsilon$.

$$\begin{aligned} |x_n y_n - x_m y_m| &= |x_n y_n - x_n y_m + x_n y_m - x_m y_m| \\ &\leq |x_n y_n - x_n y_m| + |x_n y_m - x_m y_m| \\ &= |x_n| |y_n - y_m| + |y_m| |x_n - x_m| \end{aligned}$$

$$(x_n) \ \& \ (y_n) \text{ Cauchy} \Rightarrow \begin{cases} \exists K, L \text{ s.t. } |x_n| \leq K \ \& \ |y_m| \leq L \ \forall m, n \text{ by Lemma 2.6.1} \\ \exists N_1 \text{ s.t. } n, m \geq N_1 \Rightarrow |x_n - x_m| < \frac{\varepsilon}{2L} \\ \exists N_2 \text{ s.t. } n, m \geq N_2 \Rightarrow |y_n - y_m| < \frac{\varepsilon}{2K} \end{cases}$$

Then $n, m \geq N \Rightarrow |x_n y_n - x_m y_m| \leq |x_n| |y_n - y_m| + |y_m| |x_n - x_m| < K \frac{\varepsilon}{2K} + L \frac{\varepsilon}{2L} = \varepsilon$.

2.7 Properties of Infinite Series

Theorem 2.7.1 (Algebraic Limit Theorem for Series)**.** *If* $\sum_{k=1}^{\infty} a_k = A$ *and* $\sum_{k=1}^{\infty} b_k = B$*, then*

(i) $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \mathbf{R}$ *and*

(ii) $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$.

Proof. In order to show that $\sum_{k=1}^{\infty} ca_k = cA$, we must argue that the sequence of partial sums

$$t_m = ca_1 + ca_2 + ca_3 + \cdots + ca_m$$

converges to cA. But we are given that $\sum_{k=1}^{\infty} a_k$ converges to A, meaning that the partial sums

$$s_m = a_1 + a_2 + a_3 + \cdots + a_m$$

converge to A. Because $t_m = cs_m$, applying the Algebraic Limit Theorem for sequences (Theorem 2.3.2) yields $(t_m) \to cA$, as desired.
The proof of part (ii) is analogous. □

Theorem 2.7.2 (Cauchy Criterion for Series)**.** *The series* $\sum_{k=1}^{\infty} a_k$ *converges if and only if, given* $\epsilon > 0$*, there exists an* $N \in \mathbf{N}$ *such that whenever* $n > m \geq N$ *it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Proof. Observe that

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n|$$

and apply the Cauchy Criterion for sequences. □

Theorem 2.7.3. *If the series* $\sum_{k=1}^{\infty} a_k$ *converges, then* $(a_k) \to 0$.

Proof. Consider the special case $n = m + 1$ in the Cauchy Criterion for Series. □

Theorem 2.7.4 (Comparison Test)**.** *Assume* (a_k) *and* (b_k) *are sequences satisfying* $0 \leq a_k \leq b_k$ *for all* $k \in \mathbf{N}$.

(i) *If* $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

(ii) *If* $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_k$ *diverges.*

Proof. Both statements follow immediately from the Cauchy Criterion for Series and the observation that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |b_{m+1} + b_{m+2} + \cdots + b_n|. \qquad \square$$

Example 1 (Geometric Series). A series is called geometric if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots .$$

Determine the criteria for a geometric series to converge.

If $r=1$ and $a \neq 0$, the series diverges.

If $r \neq 1$,

$$(1-r)(1+r+r^2+r^3+\cdots+r^{m-1}) = 1-r^m$$

$$\Rightarrow s_m = a+ar+ar^2+ar^3+\cdots+ar^{m-1} = \frac{a(1-r^m)}{1-r}$$

By Example 2.5.1, $r^m \to 0$ iff $|r|<1$

$$\Rightarrow \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad \text{iff } |r|<1.$$

Theorem 2.7.5 (Absolute Convergence Test). *If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.*

Proof. This proof makes use of both the necessity (the "if" direction) and the sufficiency (the "only if" direction) of the Cauchy Criterion for Series. Because $\sum_{n=1}^{\infty} |a_n|$ converges, we know that, given an $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that

$$|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \epsilon$$

for all $n > m \geq N$. By the triangle inequality,

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n|,$$

so the sufficiency of the Cauchy Criterion guarantees that $\sum_{n=1}^{\infty} a_n$ also converges. $\square$

Theorem 2.7.6 (Alternating Series Test)**.** *Let (a_n) be a sequence satisfying,*

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ *and*

(ii) $(a_n) \to 0$.

Then, the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Example 2. Proving the Alternating Series Test amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$$

converges. Different characterizations of completeness lead to different proofs.

(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.

(b) Supply another proof for this result using the Nested Interval Property (Theorem 1.4.1).

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

(a) Let $\varepsilon>0$. We need N s.t. $n>m\geq N \Rightarrow |s_n-s_m|<\varepsilon$.

$|s_n-s_m| = |a_{m+1}-a_{m+2}+a_{m+3}-\cdots\pm a_n|$

(a_n) is decreasing and $a_i>0$ $\forall i \Rightarrow |a_{m+1}-a_{m+2}+a_{m+3}-\cdots\pm a_n| \leq a_{m+1}$ $\forall n>m$ by induction

$(a_n)\to 0 \Rightarrow \exists$ N s.t. $m\geq N \Rightarrow a_m<\varepsilon$

$\Rightarrow |s_n-s_m| = |a_{m+1}-a_{m+2}+a_{m+3}-\cdots\pm a_n| \leq a_{m+1} < \varepsilon$ when $n>m\geq N$.

b) Let $I_1=[0,s_1]$. Let $I_2=[s_2,s_1]\subseteq[0,s_1]=I_1$, since (a_n) is decreasing.

Continuing in this fashion we obtain $I_1\supseteq I_2\supseteq I_3\supseteq\cdots$

By the NIP, $\exists$ at least one $S\in I_n$ $\forall$ $n\in\mathbb{N}$.

Let $\varepsilon>0$. We need N s.t. $n\geq N \Rightarrow |s_n-S|<\varepsilon$.

The length of I_n is $|s_n-s_{n-1}|=a_n$, and $(a_n)\to 0 \Rightarrow \exists$ N s.t. $n\geq N\Rightarrow a_n<\varepsilon$.

Then $|s_n-S|\leq a_n<\varepsilon$, since $s_n, S\in I_n$.

c) (s_{2n}) is increasing and bounded above (e.g. by a_1).

MCT $\Rightarrow \exists\, S \in \mathbb{R}$ s.t. $S = \lim(s_{2n})$.

Also, since $(a_n) \to 0$,

$$\lim(s_{2n+1}) = \lim(s_{2n} + a_{2n+1}) = S + \lim(a_{2n+1}) = S + 0 = S.$$

Since $(s_{2n}) \to S$ and $(s_{2n+1}) \to S$, we have $(s_n) \to S$ by Example 2.3.2.

Definition 2.7.1. If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ <u>converges absolutely</u>. If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converge, then we say that the original series $\sum_{n=1}^{\infty} a_n$ <u>converges conditionally</u>.

Example 3 (Summation-by-parts)**.** Let (x_n) and (y_n) be sequences, let $s_n = x_1 + x_2 + \cdots + x_n$ and set $s_0 = 0$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1}).$$

$$\sum_{j=m+1}^{n} x_j y_j = \sum_{j=m+1}^{n} (s_j - s_{j-1}) y_j = \sum_{j=m+1}^{n} s_j y_j - \sum_{j=m+1}^{n} s_{j-1} y_j$$

$$\sum_{j=m+1}^{n} s_{j-1} y_j = \sum_{j=m}^{n-1} s_j y_{j+1} = s_m y_{m+1} - s_n y_{n+1} + \sum_{j=m+1}^{n} s_j y_{j+1}$$

$$\sum_{j=m+1}^{n} x_j y_j = \sum_{j=m+1}^{n} s_j y_j - s_m y_{m+1} + s_n y_{n+1} - \sum_{j=m+1}^{n} s_j y_{j+1}$$

$$= s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j (y_j - y_{j+1})$$

Example 4 (Abel's Test)**.** Abel's Test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if (y_k) is a sequence satisfying

$$y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0,$$

then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Use Example 3 to show that

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}),$$

where $s_n = x_1 + x_2 + \cdots + x_n$.

(b) Use the Comparison Test to argue that $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$ converges absolutely, and show how this leads directly to a proof of Abel's Test.

(a) Let $s_n = \sum_{k=1}^{n} x_k$. Then $\exists L$ s.t. $(s_n) \to L$.

$\Rightarrow \exists M > 0$ s.t. $|s_n| \leq M \;\forall n \in \mathbb{N}$

By Ex. 3, $\forall n \in \mathbb{N}$,

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1})$$

(b) $$\sum_{k=1}^{n} |s_k (y_k - y_{k+1})| \leq \sum_{k=1}^{n} M (y_k - y_{k+1})$$

$$= M (y_1 - y_{n+1})$$

(y_{n+1}) converges $\Rightarrow \sum_{k=1}^{\infty} s_k (y_k - y_{k+1})$ converges absolutely

$\Rightarrow \sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1})$ converges

Definition 2.7.2. Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a <u>rearrangement</u> of $\sum_{k=1}^{\infty} a_k$ if there exists a one-to-one, onto function $f : \mathbf{N} \to \mathbf{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbf{N}$.

Theorem 2.7.7. *If a series converges absolutely, then any rearrangement of this series converges to the same limit.*

Proof. Assume $\sum_{k=1}^{\infty} a_k$ converges absolutely to A, and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$. Let's use

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots a_n$$

for the partial sums of the original series and use

$$t_m = \sum_{k=1}^{m} b_k = b_1 + b_2 + \cdots + b_m$$

for the partial sums of the rearranged series. Thus we want to show that $(t_m) \to A$.
Let $\epsilon > 0$. By hypothesis, $(s_n) \to A$, so choose N_1 such that

$$|s_n - A| < \frac{\epsilon}{2}$$

for all $n \geq N_1$. Because the convergence is absolute, we can choose N_2 so that

$$\sum_{k=m+1}^{n} |a_k| < \frac{\epsilon}{2}$$

for all $n > m \geq N_2$. Now, take $N = \max\{N_1, N_2\}$. We know that the finite set of terms $\{a_1, a_2, a_3, \ldots, a_N\}$ must all appear in the rearranged series, and we want to move far enough out in the series $\sum_{n=1}^{\infty} b_n$ so that we have included all of these terms. Thus, choose

$$M = \max\{f(k) : 1 \leq k \leq N\}.$$

It should now be evident that if $m \geq M$, then $(t_m - s_N)$ consists of a finite set of terms, the absolute values of which appear in the tail $\sum_{k=N+1}^{\infty} |a_k|$. Our choice of N_2 earlier then guarantees $|t_m - s_N| < \epsilon/2$, and so

$$\begin{aligned} |t_m - A| &= |t_m - s_N + s_N - A| \\ &\leq |t_m - s_N| + |s_N - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

whenever $m \geq M$. $\square$

2.8 Double Summations and Products of Infinite Series

Example 1. For $m, n \in \mathbf{N}$, set

$$s_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$$

and consider the array $\{a_{ij} : i, j \in \mathbf{N}\}$, where $a_{ij} = 1/2^{j-i}$ if $j > i$, $a_{ij} = -1$ if $j = 1$ and $a_{ij} = 0$ if $j < i$.

$$\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & \cdots \\ 0 & 0 & 0 & -1 & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Compute $\lim_{n\to\infty} s_{nn}$. Compare this to summing down rows first, and then to summing down columns first.

$s_{11} = -1$

$s_{22} = -\frac{3}{2}$

$s_{33} = -\frac{7}{4}$

$s_{nn} = -2 + \frac{1}{2^{n-1}}$

$(s_{nn}) \to -2$

$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} \right) = \sum_{i=1}^{\infty} (0) = 0$

$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ij} \right) = \sum_{j=1}^{\infty} \left(-\frac{1}{2^{j-1}} \right) = -2$

Example 2. Show that if the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges (meaning that for each fixed $i \in \mathbf{N}$ the series $\sum_{j=1}^{\infty}|a_{ij}|$ converges to some real number b_i, and the series $\sum_{i=1}^{\infty} b_i$ converges as well), then the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}$$

converges.

Fix $i \in \mathbb{N}$. $\sum_{j=1}^{\infty}|a_{ij}|$ converges $\Rightarrow \exists r_i$ s.t. $\sum_{j=1}^{\infty} a_{ij} \to r_i$.

Then $|r_i| \leq b_i$ where $b_i = \sum_{j=1}^{\infty}|a_{ij}|$.

$\sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$ converges $\Rightarrow \sum_{i=1}^{\infty}|r_i|$ converges by the Comparison Test

$\Rightarrow \sum_{i=1}^{\infty} r_i$ converges

Theorem 2.8.1. *Let $\{a_{ij} : i, j \in \mathbf{N}\}$ be a doubly indexed array of real numbers. If*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges, then both $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}$ and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}$ converge to the same value. Moreover,

$$\lim_{n\to\infty} s_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij},$$

where $s_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}$.

Proof. Define

$$t_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|$$

and let $b_i = \sum_{j=1}^{\infty} |a_{ij}|$ for all $i \in \mathbf{N}$. Our hypothesis tells us that there exists $L \geq 0$ satisfying $\sum_{i=1}^{\infty} b_i = L$. Because we are adding all non-negative terms, it follows that

$$t_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| \leq \sum_{i=1}^{m} \sum_{j=1}^{\infty} |a_{ij}| \leq \sum_{i=1}^{m} b_i \leq L.$$

Since (t_{nn}) is an increasing sequence and is bounded, it converges by the Monotone Convergence Theorem. Then since (t_{nn}) is a Cauchy sequence, given an $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that

$$|t_{nn} - t_{mm}| < \epsilon$$

for all $n > m \geq N$. Now the expression $s_{nn} - s_{mm}$ is really a sum over a finite collection of a_{ij} terms. If each a_{ij} included in the sum is replaced with $|a_{ij}|$, the sum only gets larger (this is just the triangle inequality), and the result is that

$$|s_{nn} - s_{mm}| = \left| \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} - \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \right| \leq |t_{nn} - t_{mm}| < \epsilon.$$

It follows that (s_{nn}) is Cauchy and must converge, so we can now set

$$S = \lim_{n \to \infty} s_{nn}.$$

In order to prove the theorem, we must show that the two iterated sums converge to this same limit. We will first show that

$$S = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}.$$

Because $\{t_{mn} : m, n \in \mathbf{N}\}$ is bounded above, we can let

$$B = \sup\{t_{mn} : m, n \in \mathbf{N}\}.$$

The fact that t_{mn} is a sum of non-negative terms implies that if $m_1 \geq m$ and $n_1 \geq n$ then $t_{m_1 n_1} \geq t_{mn}$. So let $N_1 = \max\{m_0, n_0\}$. Then it follows that

$$B - \frac{\epsilon}{2} < t_{m_0, n_0} \leq t_{mn} \leq B$$

for all $m, n \geq N_1$.

Without loss of generality, let $n > m \geq N$. Then,

$$\begin{aligned}
|s_{mn} - S| &= |s_{mn} - s_{mm} + s_{mm} - S| \\
&\leq |s_{mn} - s_{mm}| + |s_{mm} - S| \\
&= \left| \sum_{i=1}^{m} \sum_{j=m+1}^{n} a_{ij} \right| + |s_{mm} - S| \\
&\leq |t_{mn} - t_{mm}| + |s_{mm} - S|.
\end{aligned}$$

We have already chosen N_1 such that

$$|t_{mn} - t_{mm}| < \frac{\epsilon}{2} \quad \text{whenever } n > m \geq N_1.$$

Because $(s_{nn}) \to S$, we can pick N_2 so that

$$|s_{mm} - S| < \frac{\epsilon}{2} \quad \text{whenever } m \geq N_2.$$

Setting $N = \max\{N_1, N_2\}$, we can conclude that $|s_{mn} - S| < \epsilon/2 + \epsilon/2 = \epsilon$ for all $n > m \geq N$.

For the moment, consider $m \in \mathbf{N}$ to be fixed and write s_{mn} as

$$s_{mn} = \sum_{j=1}^{n} a_{1j} + \sum_{j=1}^{n} a_{2j} + \cdots + \sum_{j=1}^{n} a_{mj}.$$

Our hypothesis guarantees that for each fixed row i, the series $\sum_{j=1}^{\infty} a_{ij}$ converges absolutely to some real number r_i. The Algebraic Limit Theorem can then be applied to the finite number of components of s_{mn} to conclude that

$$\lim_{n \to \infty} s_{mn} = r_1 + r_2 + \cdots + r_m.$$

If, in addition, we insist that $m \geq N$, then we must have that

$$-\epsilon < s_{mn} - S < \epsilon$$

is eventually true once n is larger than N. Applying the Order Limit Theorem we find

$$-\epsilon \leq (r_1 + r_2 + \cdots + r_m) - S \leq \epsilon$$

for all $m \geq N$.

Though we have produced a "less than or equal to ϵ" result, this is not a problem. Because ϵ is arbitrary, we could just as easily have chosen to let $\epsilon' < \epsilon$ at the beginning and constructed our argument using ϵ' throughout the proof.

The same argument can be used to prove $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$ converges to S once we show that for each $j \in \mathbf{N}$ the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j.

To show $\sum_{i=1}^{\infty} a_{ij}$ converges for each $j \in \mathbf{N}$, it suffices to prove that the absolute series $\sum_{i=1}^{\infty} |a_{ij}|$ converges. Recall that $b_i = \sum_{j=1}^{\infty} |a_{ij}|$, so it is certainly the case that $b_i \geq |a_{ij}|$ for all $i, j \in \mathbf{N}$. But our hypothesis says that $\sum_{i=1}^{\infty} b_i$ converges, and so by the Comparison Test, $\sum_{i=1}^{\infty} a_{ij}$ converges for all values of j. □

Example 3. One final common way of computing a double summation is to sum along diagonals where $i + j$ equals a constant. Given a doubly indexed array $\{a_{ij} : i, j \in \mathbf{N}\}$, let

$$d_2 = a_{11}, \quad d_3 = a_{12} + a_{21}, \quad d_4 = a_{13} + a_{22} + a_{31},$$

and in general set

$$d_k = a_{1,k-1} + a_{2,k-2} + \cdots + a_{k-1,1}.$$

(a) Assuming the hypothesis—and hence the conclusion—of Theorem 2.8.1, show that $\sum_{k=2}^{\infty} d_k$ converges absolutely.

(b) Imitate the strategy in the proof of Theorem 2.8.1 to show that $\sum_{k=2}^{\infty} d_k$ converges to $S = \lim_{n\to\infty} s_{nn}$.

a) Let $u_n = |d_2| + |d_3| + |d_4| + \cdots + |d_n| = \sum_{k=2}^{n} |d_k|$.

Then $u_n = \sum_{k=2}^{n} |d_k| \leq \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}| = t_{nn}$.

(t_{nn}) converges $\Rightarrow$ u_n converges by the Comparison Test.

b) Let $\varepsilon > 0$. We need N s.t. $n \geq N \Rightarrow \left|\sum_{k=2}^{n} d_k - S\right| < \varepsilon$.

$(s_{nn}) \to S \Rightarrow \exists N_1$ s.t. $n \geq N_1 \Rightarrow |s_{nn} - S| < \frac{\varepsilon}{2}$

(t_{nn}) converges $\Rightarrow \exists N_2$ s.t. $n > m \geq N_2 \Rightarrow |t_{nn} - t_{mm}| < \frac{\varepsilon}{2}$.

Set $N = \max\{N_1, 2N_2\}$. Then, for $n \geq N$

$\left|s_{nn} - \sum_{k=2}^{n} d_k\right| \leq (t_{nn} - t_{N_2N_2}) < \frac{\varepsilon}{2}$, and so

$$\left|\sum_{k=2}^{n} d_k - S\right| = \left|\sum_{k=2}^{n} d_k - s_{nn} + s_{nn} - S\right| \leq \left|\sum_{k=2}^{n} d_k - s_{nn}\right| + |s_{nn} - S| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Remark 1. One way to carry out the algebra on a product of series is to write

$$\begin{aligned}\left(\sum_{i=1}^{\infty} a_i\right)\left(\sum_{j=1}^{\infty} b_j\right) &= (a_1 + a_2 + a_3 + \cdots)(b_1 + b_2 + b_3 + \cdots) \\ &= a_1b_1 + (a_1b_2 + a_2b_1) + (a_3b_1 + a_2b_2 + a_1b_3) + \cdots \\ &= \sum_{k=2}^{\infty} d_k,\end{aligned}$$

where

$$d_k = a_1b_{k-1} + a_2b_{k-2} + \cdots + a_{k-1}b_1.$$

This particular form of the product is called the Cauchy product of two series.

Example 4. Assume that $\sum_{i=1}^{\infty} a_i$ converges absolutely to A, and $\sum_{j=1}^{\infty} b_j$ converges absolutely to B.

(a) Show that the iterated sum $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_ib_j|$ converges so that we may apply Theorem 2.8.1.

(b) Let $s_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_j$, and prove that $\lim_{n\to\infty} s_{nn} = AB$. Conclude that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_ib_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_ib_j = \sum_{k=2}^{\infty} d_k = AB,$$

where, as before, $d_k = a_1b_{k-1} + a_2b_{k-2} + \cdots + a_{k-1}b_1$.

a) Let $\sum_{i=1}^{\infty} |a_i| = L$ and $\sum_{j=1}^{\infty} |b_j| = M$.

For fixed $i \in \mathbb{N}$, $\sum_{j=1}^{\infty} |a_ib_j| = |a_i| \sum_{j=1}^{\infty} |b_j|$, so

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_ib_j| = \sum_{i=1}^{\infty} |a_i| \sum_{j=1}^{\infty} |b_j| = \sum_{i=1}^{\infty} |a_i| M = M \sum_{i=1}^{\infty} |a_i| = ML$$

b) $$\lim_{n\to\infty} s_{nn} = \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_j = \lim_{n\to\infty}\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} b_j\right) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} a_i\right)\lim_{n\to\infty}\left(\sum_{j=1}^{n} b_j\right) = AB$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_ib_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_ib_j = \sum_{k=2}^{\infty} d_k = \lim_{n\to\infty} s_{nn} = AB$$

Chapter 3

Basic Topology of R

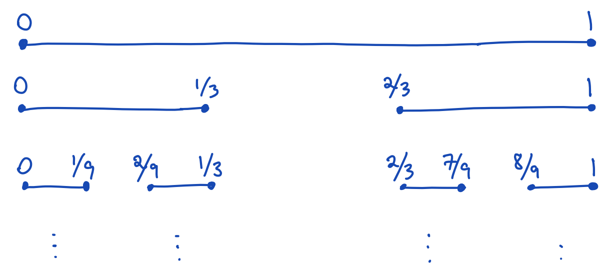
3.1 Discussion: The Cantor Set

Let $C_0 = [0,1]$.

Let $C_1 = C_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$.

Let $C_2 = \left(\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right]\right) \cup \left(\left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]\right)$.

$\vdots$



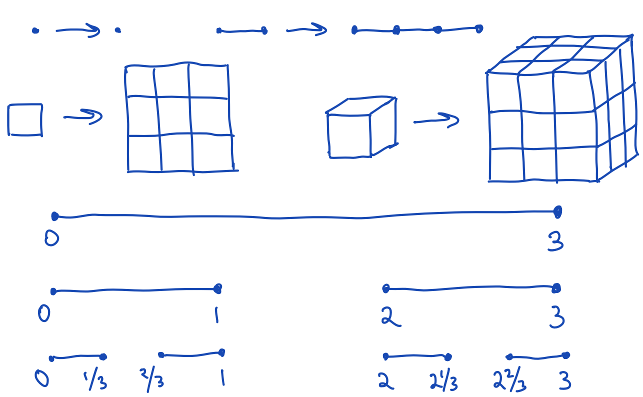
Let C_n consist of 2^n closed intervals of length $1/3^n$. The Cantor set is $C = \bigcap_{n=0}^{\infty} C_n$.

The "length" of C is $1 - \left[\frac{1}{3} + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \cdots + 2^{n-1}\left(\frac{1}{3^n}\right) + \cdots\right] = 1 - \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1 - 1 = 0$

For each $c \in C$, form (a_n) by setting $a_1 = 0$ if c is in the left part of C_1 and $a_1 = 1$ o.w.

Set $a_2 = 0$ or 1 depending on whether c is in the left or right part of C_2.

Continuing this way, every $c \in C$ corresponds to a sequence (a_n) of 0's and 1's, and vice versa, so C is uncountable.



	dim	×3	new copies
point	0	→	$1 = 3^0$
segment	1	→	$3 = 3^1$
square	2	→	$9 = 3^2$
cube	3	→	$27 = 3^3$
C	x	→	$2 = 3^x$

$\Rightarrow \dim C = \frac{\log 2}{\log 3}$

3.2 Open and Closed Sets

Definition 3.2.1. A set $O \subseteq \mathbf{R}$ is open if for all points $a \in O$ there exists an ϵ-neighborhood $V_\epsilon(a) \subseteq O$.

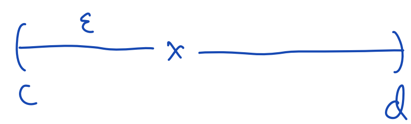
Def 2.2.3

Example 1. (i) $\mathbf{R}$ and $\emptyset$ are both open sets.

(ii) Show that the open interval

$$(c, d) = \{x \in \mathbf{R} : c < x < d\}$$

is an open set.



Let $x \in (c, d)$.

Take $\varepsilon = \min\{x-c, d-x\}$

$\Rightarrow V_\varepsilon(x) \subseteq (c, d)$

Theorem 3.2.1. (i) *The union of an arbitrary collection of open sets is open.*

(ii) *The intersection of a finite collection of open sets is open.*

Proof. To prove (i), we let $\{O_\lambda : \lambda \in \Lambda\}$ be a collection of open sets and let $O = \bigcup_{\lambda \in \Lambda} O_\lambda$. Let a be an arbitrary element of O. In order to show that O is open, Definition 3.2.1 insists that we produce an ϵ-neighborhood of a completely contained in O. But $a \in O$ implies that a is an element of at least one particular $O_{\lambda'}$. Because we are assuming $O_{\lambda'}$ is open, we can use Definition 3.2.1 to assert that there exists $V_\epsilon(a) \subseteq O_{\lambda'}$. The fact that $O_{\lambda'} \subseteq O$ allows us to conclude that $V_\epsilon(a) \subseteq O$. This completes the proof of (i).

For (ii), let $\{O_1, O_2, \ldots O_N\}$ be a finite collection of open sets. Now, if $a \in \bigcap_{k=1}^N O_k$, then a is an element of each of the open sets. By the definition of an open set, we know that, for each $1 \leq k \leq N$, there exists $V_{\epsilon_k}(a) \subseteq O_k$. Letting $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_N\}$, it follows that $V_\epsilon(a) \subseteq V_{\epsilon_k}(a)$ for all k, and hence $V_\epsilon(a) \subseteq \bigcap_{k=1}^N O_k$, as desired. □

Definition 3.2.2. A point x is a limit point of a set A if every ϵ-neighborhood $V_\epsilon(x)$ of x intersects the set A at some point other than x.

Theorem 3.2.2. *A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbf{N}$.*

Proof. ($\Rightarrow$) Assume x is a limit point of A. In order to produce a sequence (a_n) converging to x, we are going to consider the particular ϵ-neighborhoods obtained using $\epsilon = 1/n$. By Definition 3.2.2, every neighborhood of x intersects

A in some point other than x. This means that, for each $n \in \mathbf{N}$, we are justified in picking a point

$$a_n \in V_{1/n}(x) \cap A$$

with the stipulation that $a_n \neq x$. Given an arbitrary $\epsilon > 0$, choose N such that $1/N < \epsilon$. It follows that $|a_n - x| < \epsilon$ for all $n \geq N$.

($\Leftarrow$) For the reverse implication we assume $\lim a_n = x$ where $a_n \in A$ but $a_n \neq x$, and let $V_\epsilon(x)$ be an arbitrary ϵ-neighborhood. The definition of convergence assures us that there exists a term a_N in the sequence satisfying $a_N \in V_\epsilon(x)$, and the proof is complete. $\square$

Definition 3.2.3. A point $a \in A$ is an isolated point of A if it is not a limit point of A.

Definition 3.2.4. A set $F \subseteq \mathbf{R}$ is closed if it contains its limit points.

Theorem 3.2.3. *A set $F \subseteq \mathbf{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.*

Example 2. Prove Theorem 3.2.3.

($\Rightarrow$) Assume $F \subseteq \mathbb{R}$ is closed. $\Rightarrow$ F contains its limit points.
Let (a_n) be Cauchy. Then $\exists x$ s.t. $x = \lim a_n$.
If $a_n \neq x$ $\forall n$, then x is a limit point of F.
If $a_n = x$ for some n, then $(a_n) \subseteq F$, so $x \in F$.

($\Leftarrow$) Assume every Cauchy sequence in F has a limit in F. Let x be a limit point of F.
Then $x = \lim a_n$ for some (a_n). Since (a_n) converges it is Cauchy, so $x \in F$.

Example 3. (i) Consider

$$A = \left\{ \frac{1}{n} : n \in \mathbf{N} \right\}.$$

Show that each point of A is isolated.

Let $1/n \in A$ and choose $\varepsilon = 1/n - 1/(n+1)$.

Then $V_\varepsilon(1/n) \cap A = \{\frac{1}{n}\} \Rightarrow 1/n$ is not a limit point.
$\Rightarrow 1/n$ is isolated.

However, 0 is a limit point of A and $0 \notin A \Rightarrow A$ is not closed.
$A \cup \{0\}$ is closed (closure of A)

(ii) Prove that a closed interval

$$[c, d] = \{x \in \mathbf{R} : c \leq x \leq d\}$$

is a closed set using Definition 3.2.4.

If x is a limit point of $[c,d]$ then $\exists\ (x_n) \subseteq [c,d]$ s.t. $(x_n) \to x$.

$c \leq x_n \leq d \Rightarrow c \leq x \leq d$ (Order Limit Theorem)

$\Rightarrow [c,d]$ is closed

(iii) Consider the set $\mathbf{Q} \subseteq \mathbf{R}$ of rational numbers. Show that the set of limit points of $\mathbf{Q}$ is *all of* $\mathbf{R}$.

Let $y \in \mathbb{R}$. Consider $V_\varepsilon(y) = (y-\varepsilon, y+\varepsilon)$.

Theorem 1.4.3 $\Rightarrow \exists\ r \in \mathbb{Q}$ s.t. $r \neq y$ and $r \in V_\varepsilon(y)$.

$\Rightarrow y$ is a limit point of $\mathbb{Q}$.

Theorem 3.2.4 (Density of $\mathbf{Q}$ in $\mathbf{R}$). *For every $y \in \mathbf{R}$, there exists a sequence of rational numbers that converges to y.*

Proof. Combine the preceding example with Theorem 3.2.2. □

Definition 3.2.5. Given a set $A \subseteq \mathbf{R}$, let L be the set of all limit points of A. The closure of A is defined to be $\overline{A} = A \cup L$.

Example 4. Let A be nonempty and abounded above so that $s = \sup A$ exists.

(a) Show that $s \in \overline{A}$.

(b) Can an open set contain its supremum?

a) $s = \sup A \Rightarrow \exists\ \varepsilon > 0$ and $a \in A$ s.t. $s - \varepsilon < a. \Rightarrow a \in V_\varepsilon(s)$

$\Rightarrow V_\varepsilon(s)$ intersects A at a point other than $s \Rightarrow s$ is a limit point of $A \Rightarrow s \in \overline{A}$.

b) Suppose A is open, $s = \sup A$ and $s \in A$. Then $\exists\ \varepsilon$ st. $V_\varepsilon(s) \subseteq A \Rightarrow s + \frac{\varepsilon}{2} \in A$.
Contradiction.

Example 5. Given $A \subseteq \mathbf{R}$, let L be the set of all limit points of A.

(a) Show that the set L is closed.

(b) Argue that if x is a limit point of $A \cup L$, then x is a limit point of A.

(a) Let x be a limit point of L. Let $V_\varepsilon(x)$ be arbitrary.
Then $V_\varepsilon(x)$ intersects L at a point $\ell \in L$ s.t. $\ell \neq x$.
Choose ε' s.t. $V_{\varepsilon'}(\ell) \subseteq V_\varepsilon(x)$ and $x \notin V_{\varepsilon'}(\ell)$.
$\ell \in L \Rightarrow \ell$ is a limit point of A. $\Rightarrow V_{\varepsilon'}(\ell)$ intersects A.
$\Rightarrow V_\varepsilon(x)$ intersects A at a point other than x.
$\Rightarrow x$ is a limit point of $A \Rightarrow x \in L$.

b) Let $V_\varepsilon(x)$ be arbitrary. Then $V_\varepsilon(x)$ intersects $A \cup L$. Suppose $\exists\, \ell \in L$ with $\ell \in V_\varepsilon(x)$.
Choose ε' s.t. $V_{\varepsilon'}(\ell) \subseteq V_\varepsilon(x)$ and $x \notin V_{\varepsilon'}(\ell)$.
$\ell \in L \Rightarrow \ell$ is a limit point of A. $\Rightarrow V_{\varepsilon'}(\ell)$ intersects A
$\Rightarrow V_\varepsilon(x)$ intersects A at a point other than x.
$\Rightarrow x$ is a limit point of A.

Theorem 3.2.5. *For any $A \subseteq \mathbf{R}$, the closure $\overline{A}$ is a closed set and is the smallest closed set containing A.*

Proof. If L is the set of limit points of A, then it is immediately clear that $\overline{A}$ contains the limit points of A. Then since limit points of $A \cup L$ must be limit points of A by the preceding example, this shows that $\overline{A} = A \cup L$ contains its limits points and is thus closed.

Now, *any* closed set containing A must contain L as well. This shows that $\overline{A} = A \cup L$ is the smallest closed set containing A. □

Theorem 3.2.6. *A set O is open if and only if O^c is closed. Likewise, a set F is closed if and only if F^c is open.*

Proof. Given an open set $O \subseteq \mathbf{R}$, let's first prove that O^c is a closed set. To prove O^c is closed, we need to show that it contains all of its limit points. If x is a limit point of O^c, then *every* neighborhood of x contains some point of O^c. But that is enough to conclude that x cannot be in the open set O because $x \in O$ would imply that there exists a neighborhood $V_\epsilon(x) \subseteq O$. Thus, $x \in O^c$, as desired.

For the converse statement, we assume O^c is closed and argue that O is open. Thus, given an arbitrary point $x \in O$, we must produce an ϵ-neighborhood $V_\epsilon(x) \subseteq O$. Because O^c is closed, we can be sure that x is *not* a limit point of O^c. This implies there must be some neighborhood $V_\epsilon(x)$ of x that does not intersect the set O^c. But this means $V_\epsilon(x) \subseteq O$, which is precisely what we needed to show.

The second statement in the theorem follows quickly from the first using the observation that $(E^c)^c = E$ for any set $E \subseteq \mathbf{R}$. $\square$

Theorem 3.2.7. (i) *The union of a finite collection of closed sets is closed.*

(ii) *The intersection of an arbitrary collection of closed sets is closed.*

Example 6 (De Morgan's Laws)**.** (a) Given a collection of sets $\{E_\lambda : \lambda \in \Lambda\}$, show that

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c \qquad \text{and} \qquad \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c$$

(b) Now, prove Theorem 3.2.7.

a) Let $x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$. $\Rightarrow x \notin E_\lambda \ \forall \lambda$. $\Rightarrow x \in E_\lambda^c \ \forall \lambda$. $\Rightarrow x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. $\Rightarrow \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c \subseteq \bigcap_{\lambda\in\Lambda} E_\lambda^c$.

Let $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. $\Rightarrow x \notin E_\lambda \ \forall \lambda$. $\Rightarrow x \notin \bigcup_{\lambda\in\Lambda} E_\lambda \Rightarrow x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$. $\Rightarrow \bigcap_{\lambda\in\Lambda} E_\lambda^c \subseteq \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$.

$\Rightarrow \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c$.

Let $x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c$. $\Rightarrow \exists \lambda' \in \Lambda$ s.t. $x \notin E_{\lambda'} \Rightarrow x \in E_{\lambda'}^c \Rightarrow x \in \bigcup_{\lambda\in\Lambda} E_\lambda^c \Rightarrow \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda^c$.

Let $x \in \bigcup_{\lambda\in\Lambda} E_\lambda^c$. $\Rightarrow \exists \lambda' \in \Lambda$ s.t. $x \notin E_{\lambda'} \Rightarrow x \notin \bigcap_{\lambda\in\Lambda} E_\lambda \Rightarrow x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right) \Rightarrow \bigcup_{\lambda\in\Lambda} E_\lambda^c \subseteq \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c$.

$\Rightarrow \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c$.

b) i) Let E_λ be a finite collection of closed sets. $\Rightarrow E_\lambda^c$ are open.

$\Rightarrow \bigcap_{\lambda\in\Lambda} E_\lambda^c = \left(\bigcup_{\lambda\in A} E_\lambda\right)^c$ is open. $\Rightarrow \bigcup_{\lambda\in A} E_\lambda$ is closed.

ii) Let E_λ be an arbitrary collection of closed sets. $\Rightarrow E_\lambda^c$ are open.

$\Rightarrow \bigcup_{\lambda\in\Lambda} E_\lambda^c = \left(\bigcap_{\lambda\in A} E_\lambda\right)^c$ is open. $\Rightarrow \bigcap_{\lambda\in A} E_\lambda$ is closed.

3.3 Compact Sets

Definition 3.3.1 (Compactness)**.** A set $K \subseteq \mathbf{R}$ is compact if every sequence in K has a subsequence that converges to a limit that is also in K.

Example 1. Show that a closed interval is a compact set.

Let $(a_n) \subseteq [c,d]$. Then $\exists$ convergent (a_{n_k}) by the Bolzano-Weierstrass Theorem.
$[c,d]$ is closed $\Rightarrow$ the limit of (a_{n_k}) is in $[c,d]$.

Definition 3.3.2. A set $A \subseteq \mathbf{R}$ is bounded if there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$.

Theorem 3.3.1 (Characterization of Compactness in $\mathbf{R}$)**.** *A set $K \subseteq \mathbf{R}$ is compact if and only if it is closed and bounded.*

Proof. Let K be compact. We will first prove that K must be bounded, so assume, for contradiction, that K is not a bounded set. Because K is not bounded there must exist an element $x_1 \in K$ satisfying $|x_1| > 1$. Likewise, there must exist $x_2 \in K$ with $|x_2| > 2$, and in general, given any $n \in \mathbf{N}$, we can produce $x_n \in K$ such that $|x_n| > n$.

Now, because K is assumed to be compact, (x_n) should have a convergent subsequence (x_{n_k}). But the elements of the subsequence must satisfy $|x_{n_k}| > n_k$, and consequently (x_{n_k}) is unbounded. Because convergent sequences are bounded (Theorem 2.3.1), we have a contradiction. Thus, K must at least be a bounded set.

Next, we will show that K is also closed. To see that K contains its limit points, we let $x = \lim x_n$, where (x_n) is contained in K and argue that x must be in K as well. By Definition 3.3.1, the sequence (x_n) has a convergent subsequence (x_{n_k}), and by Theorem 2.5.1, we know (x_{n_k}) converges to the same limit x. Finally, Definition 3.3.1 requires that $x \in K$. This proves that K is closed.

For the converse, let $K \subseteq \mathbf{R}$ be closed and bounded. Since K is bounded, the Bolzano–Weierstrass Theorem guarantees that for any sequence (a_n) contained in K, we can find a convergent subsequence (a_{n_k}). Because the set is closed, the limit of this subsequence is also in K. Hence K is compact. $\square$

Example 2. Show that if K is compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of K.

K is compact and nonempty $\Rightarrow K$ is closed and bounded. By AoC, $\exists\, \alpha$ s.t. $\alpha = \sup K$.
$\Rightarrow \forall n \in \mathbb{N}\ \exists\, x_n \in K$ s.t. $\alpha - \frac{1}{n} < x_n \leq \alpha$. $\Rightarrow \lim x_n = \alpha$
$(x_n) \subseteq K$ and K is closed $\Rightarrow \alpha \in K$. Similarly, $\inf K \in K$.

Example 3. Decide which of the following sets are compact. For those that are not compact, show how Definition 3.3.1 breaks down. In other words, give an example of a sequence contained in the given set that does not possess a subsequence converging to a limit in the set.

(a) $\mathbf{N}$.

(b) $\mathbf{Q} \cap [0,1]$.

(c) The Cantor set.

(d) $\{1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 : n \in \mathbf{N}\}$.

(e) $\{1, 1/2, 2/3, 3/4, 4/5, \ldots\}$.

a) Not compact. Let $a_n = n$.
b) Not compact. Let a_n be a sequence converging to $1/\sqrt{2}$.
c) Compact. It is bounded and closed, since it is an infinite intersection of closed sets.
d) Not compact. All a_n converge to $\pi^2/6$.
e) Compact. Every sequence converges to 1, so it is closed and bounded.

Example 4. Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both, or neither.

(a) $K \cap F$

(b) $\overline{F^c \cup K^c}$

(c) $K \setminus F = \{x \in K : x \notin F\}$

(d) $\overline{K \cap F^c}$

a) Definitely compact. $K \cap F$ is closed and bounded, since K is bounded.
b) Definitely closed. K bounded $\Rightarrow K^c$ is unbounded
c) Both. If $K = [0,1]$ and $F^c = (0,1)$, then $K \setminus F = K \cap F^c = (0,1)$ is not closed, but if $F^c = (-1,2)$ then $K \setminus F = [0,1]$ is compact.
d) Definitely compact. K bounded $\Rightarrow K \cap F^c$ is bounded.

Theorem 3.3.2 (Nested Compact Set Property)**.** *If*

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \cdots$$

is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ *is not empty.*

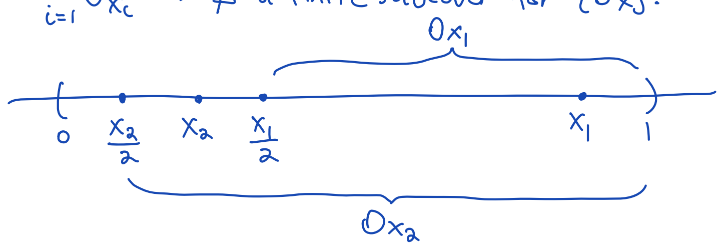
Proof. For each $n \in \mathbf{N}$, pick a point $x_n \in K_n$. Because the compact sets are nested, it follows that the sequence (x_n) is contained in K_1. By Definition 3.3.1, (x_n) has a convergent subsequence (x_{n_k}) whose limit $x = \lim x_{n_k}$ is an element of K_1.

In fact, x is an element of *every* K_n for essentially the same reason. Given a particular $n_0 \in \mathbf{N}$, the terms in the sequence (x_n) are contained in K_{n_0} as long as $n \geq n_0$. Ignoring the finite number of terms for which $n_k < n_0$, the same subsequence (x_{n_k}) is then also contained in K_{n_0}. The conclusion is that $x = \lim x_{n_k}$ is an element of K_{n_0}. Because n_0 was arbitrary, $x \in \bigcap_{n=1}^{\infty} K_n$ and the proof is complete. $\square$

Definition 3.3.3. Let $A \subseteq \mathbf{R}$. An <u>open cover</u> for A is a (possibly infinite) collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ whose union contains the set A; that is, $A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$. Given an open cover for A, a <u>finite subcover</u> is a finite subcollection of open sets from the original open cover whose union still manages to completely contain A.

Example 5. Find an open cover for the open interval $(0, 1)$, but show that it is impossible to find a finite subcover. On the other hand, find an open cover for the closed interval $[0, 1]$ that has a finite subcover.

For each $x \in (0,1)$, let O_x be the open interval $(x/2, 1)$.
Then $\{O_x : x \in (0,1)\}$ is an open cover for $(0,1)$.
Given $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$, set $x' = \min\{x_1, x_2, \ldots, x_n\}$. Then $y \in \mathbb{R}$ s.t. $0 < y \leq x'/2$ is not in $\bigcup_{i=1}^{n} O_{x_i}$ $\Rightarrow \nexists$ a finite subcover for $\{O_x\}$.



Next set $O_0 = (-\varepsilon, \varepsilon)$ and $O_1 = (1-\varepsilon, 1+\varepsilon)$ for $\varepsilon > 0$. Then $\{O_0, O_1, O_x : x \in (0,1)\}$ is an open cover for $[0,1]$. Now if we choose x' s.t. $x'/2 < \varepsilon$, then $\{O_0, O_{x'}, O_1\}$ is a finite subcover for $[0,1]$.

Theorem 3.3.3 (Heine–Borel Theorem)**.** *Let K be a subset of* $\mathbf{R}$*. All of the following statements are equivalent in the sense that any one of them implies the two others:*

(i) *K is compact.*

(ii) *K is closed and bounded.*

(iii) *Every open cover for K has a finite subcover.*

Proof. The equivalence of (i) and (ii) is the content of Theorem 3.3.1. What remains is to show that (iii) is equivalent to (i) and (ii). Let's first assume (iii), and prove that it implies (ii) (and thus (i) as well).

To show that K is bounded, we construct an open cover for K by defining O_x to be an open interval of radius 1 around each point $x \in K$. In the language of neighborhoods, $O_x = V_1(x)$. The open cover $\{O_x : x \in K\}$ then must have a finite subcover $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$. Because K is contained in a finite union of bounded sets, K must itself be bounded.

The proof that K is closed is more delicate, and we argue it by contradiction. Let (y_n) be a Cauchy sequence contained in K with $\lim y = y$. To show that K is closed, we must demonstrate that $y \in K$, so assume for contradiction that this is not the case. If $y \notin K$, then every $x \in K$ is some positive distance away from y. We now construct an open cover by taking O_x to be an interval of radius $|x - y|/2$ around each point x in K. Because we are assuming (iii), the resulting open cover $\{O_x : x \in K\}$ must have a finite subcover $\{O_{x_1}, O_{x_2}, \ldots O_{x_n}\}$. The contradiction arises when we realize that, in the spirit of the preceding example, this finite subcover cannot contain all of the elements of the sequence (y_n). To make this explicit, set

$$\epsilon_0 = \min\left\{\frac{|x_i - y|}{2} : 1 \leq i \leq n\right\}.$$

Because $(y_n) \to y$, we can certainly find a term y_N satisfying $|y_N - y| < \epsilon_0$. But such a y_N must necessarily be excluded from each O_{x_i}, meaning that

$$y_N \notin \bigcup_{i=1}^{n} O_{x_i}.$$

Thus our supposed subcover does not actually cover all of K. This contradiction implies that $y \in K$, and hence K is closed and bounded.

For the reverse implication, assume K satisfies (i) and (ii), and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K and bisect I_0 into two halves A_1 and B_1. If $A_1 \cap K$ and $B_1 \cap K$ both had finite subcovers consisting of sets from the collection $\{O_\lambda : \lambda \in \Lambda\}$, then there would exist a finite subcover for K. But we assumed that such a finite subcover did not exist for K. Hence either $A_1 \cap K$ or $B_1 \cap K$ (or both) has no finite subcover.

Let I_1 be a half of I_0 whose intersection with K does not have a finite subcover, so that $I_1 \cap K$ cannot be finitely covered and $I_1 \subseteq I_0$. Then bisect I_1 into two closed intervals, A_2 and B_2 and again let $I_2 = A_2$ if $A_2 \cap K$ does not have a finite subcover. Otherwise, let $I_2 = B_2$. Continuing this process of bisecting the interval I_n, we get a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim |I_n| = 0$. Because K is compact, $K \cap I_n$ is also compact for each $n \in \mathbf{N}$. By Theorem 3.3.2, $\bigcap_{n=1}^{\infty} I_n \cap K$ is nonempty, and there exists an $x \in K \cap I_n$ for all n.

Let $x \in K$ and let O_{λ_0} be an open set that contains x. Because O_{λ_0} is open, there exists $\epsilon_0 > 0$ such that $V_{\epsilon_0}(x) \subseteq O_{\lambda_0}$. Now choose n_0 such that $|I_{n_0}| < \epsilon_0$. Then I_{n_0} is contained in the single open set O_{λ_0} and thus it has a finite subcover. This contradiction implies that K must have originally had a finite subcover. □

Example 6. Consider each of the sets listed in Example 3. For each one that is not compact, find an open cover for which there is no finite subcover.

a) Let $O_\lambda = (\lambda-1, \lambda+1)$ for $\lambda \in \mathbb{N}$. O_λ has no finite subcover.

b) Let α be an irrational number in $(0,1)$.

For $n \in \mathbb{N}$, set $O_n = (-1, \alpha - 1/n) \cup (\alpha + 1/n, 2)$.

$\bigcup_n O_n = (-1, \alpha) \cup (\alpha, 2) \supseteq \mathbb{Q} \cap [0,1]$. O_n has no finite subcover.

d) For each $s_n = 1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2$ in the set,

let $O_n = (s_n - 1/(n+1)^2, s_n + 1/(n+1)^2)$.

O_n has no finite subcover.

3.4 Perfect Sets and Connected Sets

Definition 3.4.1. A set $P \subseteq \mathbf{R}$ is perfect if it is closed and contains no isolated points.

Example 1 (Cantor Set)**.** Show that the Cantor set is perfect.

$C = \bigcap_{n=0}^{\infty} C_n$ where each C_n is a finite union of closed intervals
$\Rightarrow$ each C_n is closed. $\Rightarrow C$ is closed.

Let $x \in C$. $x \in C_1 \Rightarrow x \in [0, 1/3]$ or $x \in [2/3, 1]$.
If $0 \le x < 1/3$, take $x_1 = 1/3$. If $x = 1/3$, take $x_1 = 0$.
If $2/3 \le x < 1$, take $x_1 = 1$. If $x = 1$, take $x_1 = 2/3$. In any case, $x_1 \in C$ with $|x - x_1| \le 1/3$.
For each $n \in \mathbb{N}$, the length of each interval in C_n is $1/3^n$.
For each n, let x_n be an endpoint of the interval that contains x. If x is an endpoint of a C_n interval, let x_n be the opposite endpoint.
$\Rightarrow x_n \in C$ with $x_n \neq x$ s.t. $|x - x_n| \le 1/3^n$ and $1/3^n \to 0 \Rightarrow (x_n) \to x$
$\Rightarrow x \in C$ is not an isolated point.

Theorem 3.4.1. *A nonempty perfect set is uncountable.*

Proof. If P is perfect and nonempty, then it must be infinite because otherwise it would consist only of isolated points. Let's assume, for contradiction, that P is countable. Thus, we can write

$$P = \{x_1, x_2, x_3, \ldots\},$$

where *every* element of P appears on this list. The idea is to construct a sequence of nested compact sets K_n, all contained in P, with the property that $x_1 \notin K_2$, $x_2 \notin K_3$, $x_3 \notin K_4$, ... Some care must be taken to ensure that each K_n is nonempty, for then we can use Theorem 3.3.2 to produce an

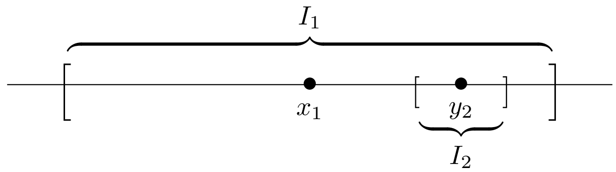
$$x \in \bigcap_{n=1}^{\infty} K_n \subseteq P$$

that *cannot* be on the list $\{x_1, x_2, x_3, \ldots\}$.

Let I_1 be a closed interval that contains x_1 in its interior (i.e., x_1 is not an endpoint of I_1). Now, x_1 is not isolated, so there exists some other point $y_2 \in P$ that is also in the interior of I_1. Construct a closed interval I_2, centered on y_2, so that $I_2 \subseteq I_1$ but $x_1 \notin I_2$. More explicitly, if $I_1 = [a, b]$, let

$$\epsilon = \min\{y_2 - a, b - y_2, |x_1 - y_2|\}.$$

Then, the interval $I_2 = [y_2 - \epsilon/2, y_2 + \epsilon/2]$ has the desired properties.



This process can be continued. Because $y_2 \in P$ is not isolated, there must exist another point $y_3 \in P$ in the interior of I_2, and we may insist that $y_3 \neq x_2$. Now, construct I_3 centered on y_3 and small enough so that $x_2 \notin I_3$ and $I_3 \subseteq I_2$. Observe that $I_3 \cap P \neq \emptyset$ because this intersection contains at least y_3.

If we carry out this construction inductively, the result is a sequence of closed intervals I_n satisfying

(i) $I_{n+1} \subseteq I_n$,

(ii) $x_n \notin I_{n+1}$, and

(iii) $I_n \cap P \neq \emptyset$.

To finish the proof, we let $K_n = I_n \cap P$. For each $n \in \mathbf{N}$, we have that K_n is closed because it is the intersection of closed sets, and bounded because it is contained in the bounded set I_n. Hence, K_n is compact. By construction, K_n is not empty and $K_{n+1} \subseteq K_n$. Thus, we can employ the Nested Compact Set Property (Theorem 3.3.2) to conclude that the intersection

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

But each K_n is a subset of P, and the fact that $x_n \notin I_{n+1}$ leads to the conclusion that $\bigcap_{n=1}^{\infty} K_n = \emptyset$, which is the sought-after contradiction. □

Definition 3.4.2. Two nonempty sets $A, B \subseteq \mathbf{R}$ are separated if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty. A set $E \subseteq \mathbf{R}$ is disconnected if it can be written as $E = A \cup B$, where A and B are nonempty separated sets.

A set that is not disconnected is called a connected set.

Example 2. (i) Verify that $E = (1, 2) \cup (2, 5)$ is disconnected.

(ii) Show that the set of rational numbers is disconnected.

(i) $[1,2] \cap (2,5) = \phi$ and $(1,2) \cap [2,5] = \phi \Rightarrow (1,2)$ and $(2,5)$ are separated.

(ii) Let $A = \mathbb{Q} \cap (-\infty, \sqrt{2})$ and $B = \mathbb{Q} \cap (\sqrt{2}, \infty)$

$A \subseteq (-\infty, \sqrt{2}) \Rightarrow$ any limit point of A is in $(-\infty, \sqrt{2}] \Rightarrow \bar{A} \cap B = \phi$

Similarly, $A \cap \bar{B} = \phi$. $\Rightarrow A$ and B are separated.

Theorem 3.4.2. *A set $E \subseteq \mathbf{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.*

Example 3. Prove Theorem 3.4.2.

($\Rightarrow$) Let E be connected and let $E = A \cup B$ for A, B disjoint, non-empty.
$\Rightarrow$ A and B are not separated. $\Rightarrow$ Either $\bar{A} \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$.
WLOG, assume $x \in \bar{A} \cap B$. Then $x \in B$ and $x \in \bar{A}$ but $x \notin A$.
$\Rightarrow$ x is a limit point of A. $\Rightarrow \exists (x_n)$ in A s.t. $(x_n) \to x$.
($\Leftarrow$) Assume $E \subseteq \mathbb{R}$ is disconnected. $\Rightarrow \exists A, B$ separated s.t. $E = A \cup B$.
WLOG, suppose $(x_n) \subseteq A$ and $(x_n) \to x$.
$\Rightarrow x \in A$ or x is a limit point of A. $\Rightarrow x \in \bar{A}$.
$\bar{A} \cup B = \emptyset \Rightarrow x \notin B$. The result follows by contrapositive.

Theorem 3.4.3. *A set $E \subseteq \mathbf{R}$ is connected if and only if whenever $a < c < b$ with $a, b \in E$, it follows that $c \in E$ as well.*

Proof. Assume E is connected, and let $a, b \in E$ and $a < c < b$. Set

$$A = (-\infty, c) \cap E \quad \text{and} \quad B = (c, \infty) \cap E.$$

Because $a \in A$ and $b \in B$, neither set is empty and, just as in Example 2 (ii), neither set contains a limit point of the other. If $E = A \cup B$, then we would have that E is disconnected, which it is not. It must then be that $A \cup B$ is missing some element of E, and c is the only possibility. Thus, $c \in E$.

Conversely, assume that E is an interval in the sense that whenever $a, b \in E$ satisfy $a < c < b$ for some c, then $c \in E$. Our intent is to use the characterization of connected sets in Theorem 3.4.2, so let $E = A \cup B$, where A and B are nonempty and disjoint. We need to show that one of these sets contains a limit point of the other. Pick $a_0 \in A$ and $b_0 \in B$, and, for the sake of the argument, assume $a_0 < b_0$. Because E is itself an interval, the interval $I_0 = [a_0, b_0]$ is contained in E. Now, bisect I_0 into two equal halves. The midpoint of I_0 must either be in A or B, and so choose $I_1 = [a_1, b_1]$ to be the half that allows us to have $a_1 \in A$ and $b_1 \in B$. Continuing this process yields a sequence of nested intervals $I_n = [a_n, b_n]$, where $a_n \in A$, $b_n \in B$, and the length $(b_n - a_n) \to 0$. By the Nested Interval Property, there exists an

$$x \in \bigcap_{n=0}^{\infty} I_n,$$

and it is straightforward to show that the sequences of endpoints each satisfy $\lim a_n = x$ and $\lim b_n = x$. But now $x \in E$ must belong to either A or B, thus making it a limit point of the other. This completes the argument. □

Example 4. A set E is totally disconnected if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A$, $y \in B$, and $E = A \cup B$.

(a) Show that $\mathbf{Q}$ is totally disconnected.

(b) Is the set of irrational numbers totally disconnected?

a) Let $x, y \in \mathbb{Q}$. Because $\mathbb{I}$ is dense in $\mathbb{R}$, we can choose $z \in \mathbb{I}$ s.t. $x < z < y$.
Let $\mathbb{Q} = A \cup B$ where $A = \mathbb{Q} \cap (-\infty, z)$, $B = \mathbb{Q} \cap (z, \infty)$.
A and B are separated by Ex. 2 (ii) and $x \in A$, $y \in B$.

b) $\mathbb{I}$ is totally disconnected because $\mathbb{Q}$ is dense in $\mathbb{R}$, so we can apply the argument in (a) by letting $x, y \in \mathbb{I}$ and choosing $z \in \mathbb{Q}$.

3.5 Baire's Theorem

Definition 3.5.1. A set $A \subseteq \mathbf{R}$ is called an $\underline{F_\sigma \text{ set}}$ if it can be written as the countable union of closed sets. A set $B \subseteq \mathbf{R}$ is called a $\underline{G_\delta \text{ set}}$ if it can be written as the countable intersection of open sets.

Example 1. Argue that a set A is a G_δ set if and only if its complement is an F_σ set.

($\Rightarrow$) Let A be a G_δ set. $\Rightarrow A = \bigcap_{n=1}^{\infty} O_n$ where O_n is open.

$\Rightarrow A^c = \bigcup_{n=1}^{\infty} O_n^c$ where O_n^c is closed.

$\Rightarrow A^c$ is a F_σ set.

($\Leftarrow$) Let B be an F_σ set. $\Rightarrow B = \bigcup_{n=1}^{\infty} F_n$ where F_n is closed.

$\Rightarrow B^c = \bigcap_{n=1}^{\infty} F_n^c$ where F_n^c is open.

$\Rightarrow B^c$ is a G_δ set.

Example 2. Replace each ____________ with the word *finite* or *countable*, depending on which is more appropriate.

(a) The countable union of F_σ sets is an F_σ set.

(b) The finite intersection of F_σ sets is an F_σ set.

(c) The finite union of G_δ sets is a G_δ set.

(d) The countable intersection of G_δ sets is a G_δ set.

Example 3. (a) Show that a closed interval $[a, b]$ is a G_δ set.

(b) Show that the half-open interval $(a, b]$ is both a G_δ and an F_σ set.

(c) Show that $\mathbf{Q}$ is an F_σ set, and the set of irrationals $\mathbf{I}$ forms a G_δ set.

(a) $[a,b] = \bigcap_{n=1}^{\infty} (a-1/n, b+1/n)$

(b) $(a,b] = \bigcap_{n=1}^{\infty} (a, b+1/n)$; $(a,b] = \bigcup_{n=1}^{\infty} [a+1/n, b]$

(c) $\mathbb{Q}$ is countable $\Rightarrow \mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$ where $\{r_n\}$ is closed, so $\{r_n\}^c$ is open.

$\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\} \Rightarrow \mathbb{Q}$ is F_σ

$\mathbb{I} = \mathbb{Q}^c = \bigcap_{n=1}^{\infty} \{r_n\}^c \Rightarrow \mathbb{I}$ is G_δ

Theorem 3.5.1. *If $\{G_1, G_2, G_3, \ldots\}$ is a countable collection of dense, open sets, then the intersection $\bigcap_{n=1}^{\infty} G_n$ is not empty.*

Proof. Pick a point $x_1 \in G_1$. Since G_1 is open, there exists an $\epsilon_1 > 0$ such that $V_{\epsilon_1}(x_1) \subseteq G_1$. Now take $\epsilon_1' < \epsilon_1$, and let

$$I_1 = \overline{V_{\epsilon_1'}(x_1)}.$$

The significant point to make here is that I_1 is a closed interval but we still have the containment $I_1 \subseteq V_{\epsilon_1}(x_1) \subseteq G_1$.

Because G_2 is dense, there exists an $x_2 \in V_{\epsilon_1'}(x_1) \subseteq G_1$. Now $G_2 \cap V_{\epsilon_1'}(x_1)$ is open, so there exists an $\epsilon_2 > 0$ such that $V_{\epsilon_2}(x_2) \subseteq G_2 \cap V_{\epsilon_1'}(x_1)$. If we again choose a smaller $\epsilon_2' < \epsilon_2$, then as before the closed interval

$$I_2 = \overline{V_{\epsilon_2'}(x_2)}$$

satisfies $I_2 \subseteq G_2$ as well as $I_2 \subseteq I_1$. We may continue this process to create a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ satisfying $I_n \subseteq G_n$ for all $n \in \mathbf{N}$.

By the Nested Interval Property, there exists an $x \in \bigcap_{n=1}^{\infty} I_n$. Because $I_n \subseteq G_n$ it follows that $x \in G_n$ for all n. Hence $\bigcap_{n=1}^{\infty} G_n$ is not empty. □

Example 4. Show that it is impossible to write

$$\mathbf{R} = \bigcup_{n=1}^{\infty} F_n,$$

where for each $n \in \mathbf{N}$, F_n is a closed set containing no nonempty open intervals.

Let F be a closed set containing no nonempty open intervals. $\Rightarrow F^c$ is open.

Let $x, y \in \mathbb{R}$ s.t. $x < y$. Then $(x,y) \not\subseteq F$.

$\Rightarrow \exists\, z \in F^c$ s.t. $x < z < y$

$\Rightarrow F^c$ is dense.

Assume $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$ where each F_n is a closed set containing no nonempty open intervals.

$\Rightarrow \phi = \bigcap_{n=1}^{\infty} F_n^c$ where each F_n^c is a dense, open set.

Contradiction.

Example 5. Show how the previous example implies that the set $\mathbf{I}$ of irrationals cannot be an F_σ set, and $\mathbf{Q}$ cannot be a G_δ set.

Assume $\mathbb{I}$ is F_σ. Then $\mathbb{I} = \bigcup_{n=1}^{\infty} F_n$, where each F_n is closed. $F_n \subseteq \mathbb{I}$.

$\Rightarrow F_n$ contains no open intervals.

$\mathbb{Q} = \bigcup_{n=1}^{\infty} r_n$ where r_n is closed and contains no open intervals.

$\Rightarrow \mathbb{R}$ can be written as a countable union of closed sets containing no open intervals. Contradiction.

If $\mathbb{Q}$ was G_δ, then $\mathbb{I}$ would be F_σ.

Definition 3.5.2. A set E is nowhere-dense if $\overline{E}$ contains no nonempty open intervals.

Example 6. Show that a set E is nowhere-dense in $\mathbf{R}$ if and only if the complement of $\overline{E}$ is dense in $\mathbf{R}$.

($\Rightarrow$) Assume E is nowhere-dense in $\mathbb{R}$.

$\Rightarrow$ Given $x, y \in \mathbb{R}$ with $x<y$, $(x,y) \notin \overline{E}$. $\Rightarrow \exists z \in \overline{E}^c$ s.t. $x<z<y$

$\Rightarrow \overline{E}^c$ is dense.

($\Leftarrow$) Assume $\overline{E}^c$ is dense. Then $\forall x,y \in \mathbb{R}$ with $x<y$ $\exists z \in \overline{E}^c$ s.t. $x<z<y$.

$\Rightarrow \overline{E}$ cannot contain any nonempty open intervals.

Example 7. Decide whether the following sets are dense in $\mathbf{R}$, nowhere-dense in $\mathbf{R}$, or somewhere in between.

(a) $A = \mathbf{Q} \cap [0, 5]$. Somewhere in between.

(b) $B = \{1/n : n \in \mathbf{N}\}$. Nowhere dense.

(c) the set of irrationals. Dense.

(d) the Cantor set. Nowhere dense.

Theorem 3.5.2 (Baire's Theorem)**.** *The set of real numbers* $\mathbf{R}$ *cannot be written as the countable union of nowhere-dense sets.*

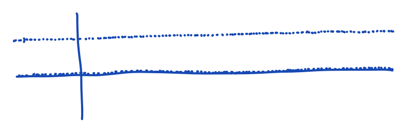
Proof. For contradiction, assume that $E_1, E_2, E_3, \ldots$ are each nowhere-dense and satisfy $\mathbf{R} = \bigcup_{n=1}^{\infty} E_n$. Then certainly $\mathbf{R} = \bigcup_{n=1}^{\infty} \overline{E_n}$. By De Morgan's Law this implies that $\emptyset = \bigcap_{n=1}^{\infty} \overline{E_n}^c$. Because E_n is nowhere dense, $\overline{E_n}^c$ is dense. We also know that $\overline{E_n}^c$ is open. Then this is a contradiction, since by Theorem 3.5.1 the countable intersection of dense, open sets is not empty. $\square$

Chapter 4

Functional Limits and Continuity

4.1 Discussion: Examples of Dirichlet and Thomae

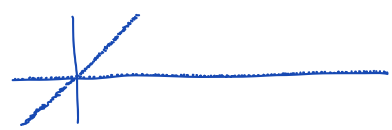
Dirichlet function $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$



If (x_n) is rational, $\lim\limits_{n\to\infty} g(x_n) = 1$, and if (x_n) is irrational, $\lim\limits_{n\to\infty} g(x_n) = 0$

For any $z \in \mathbb{R}$, we can find $(x_n) \subseteq \mathbb{Q}$ and $(y_n) \subseteq \mathbb{I}$ s.t. $\lim x_n = \lim y_n = z$, but $\lim g(x_n) \neq g(y_n)$, so g is nowhere-continuous

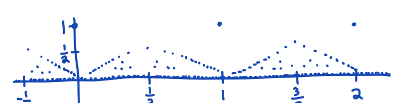
Modified Dirichlet Function $h(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$



For any $c \neq 0$ we can find $(x_n) \subseteq \mathbb{Q}$ and $(y_n) \subseteq \mathbb{I}$ s.t. $\lim x_n = \lim y_n = c$ and $\lim h(x_n) = c$ and $\lim h(y_n) = 0$, so g is not continuous at every $c \neq 0$

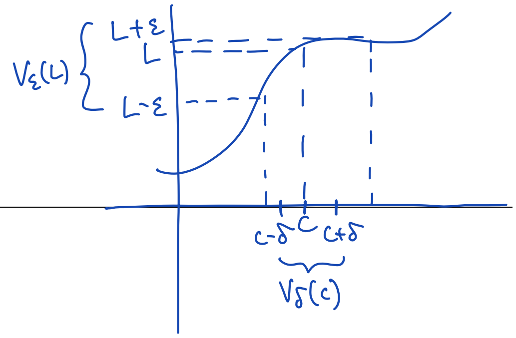
So we want $\lim\limits_{x\to c} h(x) = L$ to imply $h(z_n) \to L \ \forall (z_n) \to c$

Thomae's function $t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbb{Q}\setminus\{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$



For $c \in \mathbb{Q}$, we can find (y_n) in $\mathbb{I}$ s.t. $(y_n) \to c$ but $\lim t(y_n) = 0 \neq t(c)$, so $t(x)$ is not continuous for $c \in \mathbb{Q}$

For $c \in \mathbb{I}$, we can find (x_n) in $\mathbb{Q}$ s.t. $(x_n) \to c$ and then $\lim t(y_n) = 0 = t(c)$, so $t(x)$ is continuous for $c \in \mathbb{I}$



4.2 Functional Limits

Definition 4.2.1 (Functional Limit)**.** Let $f : A \to \mathbf{R}$, and let c be a limit point of the domain A. We say that $\lim_{x\to c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

Definition 4.2.1B (Functional Limit: Topological Version)**.** Let c be a limit point of the domain of $f : A \to \mathbf{R}$. We say $\lim_{x\to c} f(x) = L$ provided that, for every ϵ-neighborhood $V_\epsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ around c with the property that for all $x \in V_\delta(c)$ different from c (with $x \in A$) it follows that $f(x) \in V_\epsilon(L)$.

Example 1. (i) Prove that if $f(x) = 3x + 1$, then

$$\lim_{x\to 2} f(x) = 7.$$

(ii) Show that if $g(x) = x^2$, then

$$\lim_{x\to 2} g(x) = 4.$$

(i) Let $\varepsilon > 0$. We need $\delta > 0$ s.t. if $0 < |x-2| < \delta$ then $|f(x) - 7| < \varepsilon$

$$|f(x) - 7| = |(3x+1) - 7| = |3x - 6| = 3|x-2|$$

Choose $\delta = \varepsilon/3$. Then $0 < |x-2| < \delta$ implies $|f(x) - 7| < 3(\varepsilon/3) = \varepsilon$

(ii) Let $\varepsilon > 0$. We need $\delta > 0$ s.t. if $0 < |x-2| < \delta$ then $|g(x) - 4| < \varepsilon$

$$|g(x) - 4| = |x^2 - 4| = |x+2||x-2|$$

If $\delta = 1$, then $|x+2| \le |3+2| = 5 \;\forall\, x \in V_\delta(c)$

Choose $\delta = \min\{1, \varepsilon/5\}$.

Then $0 < |x-2| \delta$ implies $|x^2 - 4| = |x+2||x-2| < (5)\frac{\varepsilon}{5} = \varepsilon$

Theorem 4.2.1 (Sequential Criterion for Functional Limits). *Given a function $f : A \to \mathbf{R}$ and a limit point c of A, the following two statements are equivalent:*

(i) $\lim_{x \to c} f(x) = L$.

(ii) *For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$.*

Proof. ($\Rightarrow$) Let's first assume that $\lim_{x\to c} f(x) = L$. To prove (ii), we consider an arbitrary sequence (x_n), which converges to c and satisfies $x_n \neq c$. Our goal is to show that the image sequence $f(x_n)$ converges to L. This is most easily seen using the topological formulation of the definition.

Let $\epsilon > 0$. Because we are assuming (i), Definition 4.2.1B implies that there exists $V_\delta(c)$ with the property that all $x \in V_\delta(c)$ different from c satisfy $f(x) \in V_\epsilon(L)$. All we need to do then is argue that our particular sequence (x_n) is eventually in $V_\delta(c)$. But we are assuming that $(x_n) \to c$. This implies that there exists a point x_N after which $x_n \in V_\delta(c)$. It follows that $n \geq N$ implies $f(x_n) \in V_\epsilon(L)$, as desired.

($\Leftarrow$) For this implication we give a contrapositive proof, which is essentially a proof by contradiction. Thus, we assume that statement (ii) is true, and carefully negate statement (i). To say that

$$\lim_{x \to c} f(x) \neq L$$

means that there exists at least one particular $\epsilon_0 > 0$ for which no δ is a suitable response. In other words, no matter what $\delta > 0$ we try, there will always be at least one point

$$x \in V_\delta(c) \quad \text{with} \quad x \neq c \quad \text{for which } f(x) \notin V_{\epsilon_0}(L).$$

Now consider $\delta_n = 1/n$. From the preceding discussion, it follows that for each $n \in \mathbf{N}$ we may pick an $x_n \in V_{\delta_n}(c)$ with $x_n \neq c$ and $f(x_n) \notin V_{\epsilon_0}(L)$. But now notice that the result of this is a sequence $(x_n) \to c$ with $x_n \neq c$, where the image sequence $f(x_n)$ certainly does *not* converge to L.

Because this contradicts (ii), which we are assuming is true for this argument, we may conclude that (i) must also hold. □

Corollary 4.2.1 (Algebraic Limit Theorem for Functional Limits). *Let f and g be functions defined on a domain $A \subseteq \mathbf{R}$, and assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ for some limit point c of A. Then,*

(i) $\lim_{x \to c} kf(x) = kL$ *for all* $k \in \mathbf{R}$,

(ii) $\lim_{x \to c}[f(x) + g(x)] = L + M$,

(iii) $\lim_{x \to c}[f(x)g(x)] = LM$,

(iv) $\lim_{x \to c} f(x)/g(x) = L/M$, *provided* $M \neq 0$.

Example 2. Prove Corollary 4.2.1.

(i) $f(x_n) \to L$ when $(x_n) \to c$

$\Rightarrow kf(x_n) \to kL \ \forall k \in \mathbb{R}$ when $(x_n) \to c$ by the Algebraic Limit Theorem for sequences

$\Rightarrow \lim_{x \to c} kf(x) = kL \ \forall k \in \mathbb{R}$

(ii) $f(x_n) \to L$ and $g(x_n) \to M$ when $(x_n) \to c$

$\Rightarrow f(x_n) + g(x_n) \to L + M$ when $(x_n) \to c$ by the Algebraic Limit Theorem for sequences

$\Rightarrow \lim_{x \to c} [f(x) + g(x)] = L + M$

(iii) $f(x_n) \to L$ and $g(x_n) \to M$ when $(x_n) \to c$

$\Rightarrow f(x_n) g(x_n) \to LM$ when $(x_n) \to c$ by the Algebraic Limit Theorem for sequences

$\Rightarrow \lim_{x \to c} [f(x) g(x)] = LM$

(iv) $f(x_n) \to L$ and $g(x_n) \to M$ when $(x_n) \to c$

$\Rightarrow f(x_n)/g(x_n) \to L/M$ when $(x_n) \to c$ by the Algebraic Limit Theorem for sequences

$\Rightarrow \lim_{x \to c} [f(x)/g(x)] = L/M$

Corollary 4.2.2 (Divergence Criterion for Functional Limits)**.** *Let* f *be a function defined on* A*, and let* c *be a limit point of* A*. If there exist two sequences* (x_n) *and* (y_n) *in* A *with* $x_n \neq c$ *and* $y_n \neq c$ *and*

$$\lim x_n = \lim y_n = c \quad \textit{but} \ \lim f(x_n) \neq \lim f(y_n),$$

then we can conclude that the functional limit $\lim_{x \to c} f(x)$ *does not exist.*

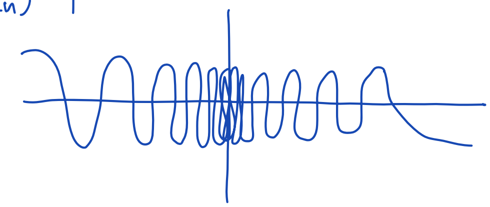
Example 3. Assuming the familiar properties of the sine function, show that $\lim_{x\to 0}\sin(1/x)$ does not exist.

If $x_n = 1/2n\pi$ and $y_n = 1/(2n\pi + \pi/2)$, then $\lim(x_n) = \lim(y_n) = 0$

But $\sin(1/x_n) = 0 \ \forall n \in \mathbb{N}$ while $\sin(1/y_n) = 1$

$\Rightarrow \lim \sin(1/x_n) \neq \lim \sin(1/y_n)$

$\Rightarrow \lim_{x\to 0} \sin(1/x)$ DNE



Example 4 (Infinite Limits)**.** *Definition:* $\lim_{x\to c} f(x) = \infty$ means that for all $M > 0$ we can find a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, it follows that $f(x) > M$.

(a) Show $\lim_{x\to 0} 1/x^2 = \infty$ in the sense described in the previous definition.

(b) Now, construct a definition for the statement $\lim_{x\to\infty} f(x) = L$. Show $\lim_{x\to\infty} 1/x = 0$.

(c) What would a rigorous definition for $\lim_{x\to\infty} f(x) = \infty$ look like? Given an example of such a limit.

(a) Let $M>0$. Choose $\delta = \sqrt{\frac{1}{M}}$. Then $0<|x|<\delta = \sqrt{\frac{1}{M}} \Rightarrow x^2 < \frac{1}{M} \Rightarrow \frac{1}{x^2} > M$.

(b) $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon>0$ there exists $K>0$ s.t. when $x>K$ it follows that $|f(x)-L|<\varepsilon$.

Let $\varepsilon>0$. Choose $K = 1/\varepsilon$. If $x>K=1/\varepsilon$, then $1/x<\varepsilon$.

(c) $\lim_{x\to\infty} f(x)=\infty$ if for every $M>0$ there exists $K>0$ s.t. when $x>K$ it follows that $f(x)>M$.

An example is $f(x)=\sqrt{x}$. Let $M>0$. Choose $K=M^2$. If $x>K=M^2$, then $\sqrt{x}>M$.

Example 5 (Squeeze Theorem)**.** Let f, g, and h satisfy $f(x) \le g(x) \le h(x)$ for all x in some common domain A. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ at some limit point c of A, show $\lim_{x\to c} g(x) = L$ as well.

Let $\varepsilon>0$. $\lim_{x\to c} f(x) = L \Rightarrow \exists\ \delta_1>0$ s.t. $0<|x-c|<\delta_1$ implies $L-\varepsilon<h(x)<L+\varepsilon$.

Choose $\delta=\min\{\delta_1,\delta_2\}$. Then $L-\varepsilon < f(x) \le g(x) \le h(x) < L+\varepsilon$ when $0<|x-c|<\delta$.

$\Rightarrow |g(x)-L|<\varepsilon$.

4.3 Continuous Functions

Definition 4.3.1 (Continuity)**.** A function $f : A \to \mathbf{R}$ is continuous at a point $c \in A$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$.

If f is continuous at every point in the domain A, then we say that f is continuous on A.

Theorem 4.3.1 (Characterizations of Continuity)**.** *Let $f : A \to \mathbf{R}$, and let $c \in A$. The function f is continuous at c if and only if any one of the following three conditions is met:*

(i) *For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies $|f(x) - f(c)| < \epsilon$;*

(ii) *For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implies $f(x) \in V_\epsilon(f(c))$;*

(iii) *If $(x_n) \to c$ (with $x_n \in A$), then $f(x_n) \to f(c)$.*

If c is a limit point of A, then the above conditions are equivalent to

(iv) $\lim_{x \to c} f(x) = f(c)$.

Proof. Statement (i) is just Definition 4.3.1, and statement (ii) is the standard rewording of (i) using topological neighborhoods in place of the absolute value notation. Statement (iii) is equivalent to (i) via an argument nearly identical to that of Theorem 4.2.1, with some slight modifications for when $x_n = c$. Finally, statement (iv) is seen to be equivalent to (i) by considering Definition 4.2.1 and observing that the case $x = c$ (which is excluded in the definition of functional limits) leads to the requirement $f(c) \in V_\epsilon(f(c))$, which is trivially true. $\square$

Corollary 4.3.1 (Criterion for Discontinuity)**.** *Let $f : A \to \mathbf{R}$, and let $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \to c$ but such that $f(x_n)$ does not converge to $f(c)$, we may conclude that f is not continuous at c.*

Theorem 4.3.2 (Algebraic Continuity Theorem)**.** *Assume $f : A \to \mathbf{R}$ and $g : A \to \mathbf{R}$ are continuous at a point $c \in A$. Then,*

(i) *$kf(x)$ is continuous at c for all $k \in \mathbf{R}$;*

(ii) *$f(x) + g(x)$ is continuous at c;*

(iii) *$f(x)g(x)$ is continuous at c; and*

(iv) *$f(x)/g(x)$ is continuous at c, provided the quotient is defined.*

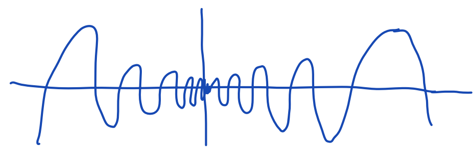
Proof. All of these statements can be quickly derived from Corollary 4.2.1 and Theorem 4.3.1. □

Example 1. Show that polynomials are continuous on **R** and that rational functions (i.e., quotients of polynomials) are continuous wherever they are defined.

Let $g(x) = x$. Because $|g(x) - g(c)| = |x - c|$ we can choose $\delta = \varepsilon$ for a given $\varepsilon > 0$.
$\Rightarrow g$ is continuous on $\mathbb{R}$.
$f(x) = k$ is also continuous by letting $\delta = 1$ for any $\varepsilon > 0$.
$\Rightarrow p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is continuous because it consists of sums and products of continuous functions.
Similarly, quotients of polynomials are continuous when the denominator is not 0.

Example 2. Investigate the continuity of

$$g(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$



$|g(x) - g(0)| = |x \sin(1/x) - 0| \leq |x|$
Given $\varepsilon > 0$, set $\delta = \varepsilon$.
$\Rightarrow$ when $|x - 0| = |x| < \delta$ it follows that $|g(x) - 0| < \varepsilon$
$\Rightarrow g$ is continuous at the origin

Example 3. Investigate the continuity of the greatest integer function $h(x) = [[x]]$ which for each $x \in \mathbf{R}$ returns the largest integer $n \in \mathbf{Z}$ satisfying $n \leq x$.

Given $m \in \mathbb{Z}$, define (x_n) by $x_n = m - 1/n$. $\Rightarrow (x_n) \to m$
But $h(x_n) \to (m-1) \neq m = h(m) \Rightarrow h$ is not continuous at $m \in \mathbb{Z}$
Let $\varepsilon > 0$. We have for any $c \in \mathbb{R}, c \notin \mathbb{Z}$, $n < c < n+1$ for some $n \in \mathbb{Z}$.
Take $\delta = \min\{c - n, (n+1) - c\}$. Then $h(x) = h(c)\ \forall\ x \in V_\delta(c)$.
$\Rightarrow h(x) \in V_\varepsilon(h(c))$ whenever $x \in V_\delta(c)$

Example 4. Consider $f(x) = \sqrt{x}$ defined on $A = \{x \in \mathbf{R} : x \geq 0\}$. Prove that f is continuous on A.

Let $\varepsilon > 0$.

If $c = 0$, then $|f(x) - f(c)| = \sqrt{x} < \varepsilon$ when $x < \varepsilon^2$, so choose $\delta = \varepsilon^2$.

Then $|x - 0| < \delta$ implies $|f(x) - 0| < \varepsilon$.

Let $c \in A$ s.t. $c \neq 0$. Then

$$|\sqrt{x} - \sqrt{c}| = |\sqrt{x} - \sqrt{c}|\left(\frac{\sqrt{x}+\sqrt{c}}{\sqrt{x}+\sqrt{c}}\right) = \frac{|x-c|}{\sqrt{x}+\sqrt{c}} \leq \frac{|x-c|}{\sqrt{c}}$$

Choose $\delta = \varepsilon\sqrt{c}$. Then $|x-c| < \delta$ implies

$$|\sqrt{x} - \sqrt{c}| < \frac{\varepsilon\sqrt{c}}{\sqrt{c}} = \varepsilon.$$

Theorem 4.3.3 (Composition of Continuous Functions)**.** *Given $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f(x) = g(f(x)$ is defined on A.*

If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

Example 5. Prove Theorem 4.3.3.

Let $\varepsilon > 0$. g is continuous at $f(c) \in B$

$\Rightarrow$ given $\varepsilon > 0$, $\exists\, \alpha > 0$ s.t. $|g(y) - g(f(c))| < \varepsilon$ when y satisfies $|y - f(c)| < \alpha$

f is continuous at $c \in A$

$\Rightarrow$ we can find $\delta > 0$ s.t. $|x - c| < \delta$ implies $|f(x) - f(c)| < \delta$

$\Rightarrow$ for $\varepsilon > 0$ $\exists\, \delta > 0$ s.t. $|x-c| < \delta$ implies $|g(f(x)) - g(f(c))| < \varepsilon$.

$\Rightarrow$ $g \circ f$ is continuous at c.

Assume $(x_n) \to c$ (with $c \in A$).

f is continuous at $c \Rightarrow f(x_n) \to f(c)$

g is continuous at $f(c) \Rightarrow g(f(x_n)) \to g(f(c))$

$\Rightarrow$ $g \circ f$ is continuous at c.

4.4 Continuous Functions on Compact Sets

Theorem 4.4.1 (Preservation of Compact Sets)**.** *Let $f : A \to \mathbf{R}$ be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is compact as well.*

Proof. Let (y_n) be an arbitrary sequence contained in the range set $f(K)$. To assert that $(y_n) \subseteq f(K)$ means that, for each $n \in \mathbf{N}$, we can find (at least one) $x_n \in K$ with $f(x_n) = y_n$. This yields a sequence $(x_n) \subseteq K$. Because K is compact, there exists a convergent subsequence (x_{n_k}) whose limit $x = \lim x_{n_k}$ is also in K. Finally, we make use of the fact that f is assumed to be continuous on A and so is continuous at x in particular. Given that $(x_{n_k}) \to x$, we conclude that $(y_{n_k}) \to f(x)$. Because $x \in K$, we have that $f(x) \in f(K)$, and hence $f(K)$ is compact. □

Theorem 4.4.2 (Extreme Value Theorem)**.** *If $f : K \to \mathbf{R}$ is continuous on a compact set $K \subseteq \mathbf{R}$, then f attains a maximum and minimum value. In other words, there exist $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.*

Proof. Because $f(K)$ is compact, we can set $\alpha = \sup f(K)$ and know $\alpha \in f(K)$. It follows that there exist $x_1 \in K$ with $\alpha = f(x_1)$. The argument for the minimum value is similar. □

Example 1. (i) Show directly that $f(x) = 3x + 1$ is continuous on $\mathbf{R}$.

(ii) Show directly that $g(x) = x^2$ is continuous on $\mathbf{R}$.

(i) Let $\varepsilon > 0$. $|f(x) - f(c)| = |(3x+1) - (3c+1)| = 3|x-c|$ for $c \in \mathbb{R}$

Choose $\delta = \varepsilon/3$. Then $|x-c| < \delta$ implies

$$|f(x) - f(c)| = 3|x-c| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

(ii) Given $c \in \mathbb{R}$, $|g(x) - g(c)| = |x^2 - c^2| = |x-c||x+c|$

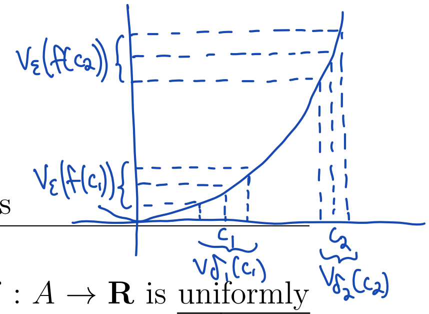
If $\delta = 1$, then $|x+c| \leq |x| + |c| \leq (|c|+1) + |c| = 2|c| + 1$

Let $\varepsilon > 0$ and choose $\delta = \min\{1, \varepsilon/2|c|+1\}$.

Then $|x-c| < \delta$ implies

$$|f(x) - f(c)| = |x-c||x+c| < \left(\frac{\varepsilon}{2|c|+1}\right)(2|c|+1) = \varepsilon.$$

Note that here δ depends on c.



Definition 4.4.1 (Uniform Continuity)**.** A function $f : A \to \mathbf{R}$ is uniformly continuous on A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Theorem 4.4.3 (Sequential Criterion for Absence of Uniform Continuity)**.** *A function $f : A \to \mathbf{R}$ fails to be uniformly continuous on A if and only if there exists a particular $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A satisfying*

$$|x_n - y_n| \to 0 \quad \textit{but} \quad |f(x_n) - f(y_n)| \geq \epsilon_0.$$

Proof. The negation of Definition 4.4.1 states that f is not uniformly continuous on A if and only if there exists $\epsilon_0 > 0$ such that for all $\delta > 0$ we can find two points x and y satisfying $|x - y| < \delta$ but with $|f(x) - f(y)| \geq \epsilon_0$. Thus, if we set $\delta_1 = 1$, then there exist two points x_1 and y_1 where $|x_1 - y_1| < 1$ but $|f(x_1) - f(y_1)| \geq \epsilon_0$.

In a similar way, if we set $\delta_n = 1/n$ where $x \in \mathbf{N}$, it follows that there exist points x_n and y_n with $|x_n - y_n| < 1/n$ but where $|f(x_1) - f(y_1)| \geq \epsilon_0$. The resulting sequences (x_n) and (y_n) satisfy the requirements described in the theorem.

Conversely, if ϵ_0, (x_n) and (y_n) exist as described, it is straightforward to see that no $\delta > 0$ is a suitable response for ϵ_0. □

Example 2. Show that $h(x) = \sin(1/x)$ is not uniformly continuous on $(0, 1)$.

Take $\varepsilon_0 = 2$ and set $x_n = \dfrac{1}{\pi/2 + 2n\pi}$ and $y_n = \dfrac{1}{3\pi/2 + 2n\pi}$

Then $x_n \to 0$, $y_n \to 0 \Rightarrow |x_n - y_n| \to 0$

But $|h(x_n) - h(y_n)| = 2 \quad \forall\ n \in \mathbb{N}$.

Example 3. Show that $f(x) = 1/x^2$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

$$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2y^2}\right| = |y - x|\left(\frac{y + x}{x^2y^2}\right)$$

If $x, y \geq 1$, then $\dfrac{y+x}{x^2y^2} = \dfrac{1}{x^2y} + \dfrac{1}{xy^2} \leq 1 + 1 = 2$

So given $\varepsilon > 0$, choose $\delta = \varepsilon/2$ and then $|f(x) - f(y)| < (\varepsilon/2)2 = \varepsilon$ when $|x - y| < \delta$

$\Rightarrow f$ is uniformly continuous on $[1, \infty)$

On $(0, 1]$, set $x_n = 1/\sqrt{n}$ and $y_n = 1/\sqrt{n+1}$. Then $|x_n - y_n| \to 0$

but $|f(x_n) - f(y_n)| = |n - (n+1)| = 1$

Theorem 4.4.4 (Uniform Continuity on Compact Sets)**.** *A function that is continuous on a compact set K is uniformly continuous on K.*

Proof. Assume $f : K \to \mathbf{R}$ is continuous at every point of a compact set $K \subseteq \mathbf{R}$. To prove that f is uniformly continuous on K we argue by contradiction.

By the criterion in Theorem 4.4.3, if f is not uniformly continuous on K, then there exist two sequences (x_n) and (y_n) in K such that

$$\lim |x_n - y_n| = 0 \quad \text{while} \quad |f(x_n) - f(y_n)| \geq \epsilon_0$$

for some particular $\epsilon_0 > 0$. Because K is compact, the sequence (x_n) has a convergent subsequence (x_{n_k}) with $x = \lim x_{n_k}$ also in K.

Next consider the subsequence (y_{n_k}) consisting of those terms in (y_n) that correspond to the terms in the convergent subsequence (x_{n_k}). By the Algebraic Limit Theorem,

$$\lim(y_{n_k}) = \lim((y_{n_k} - x_{n_k}) + x_{n_k}) = 0 + x.$$

The conclusion is that both (x_{n_k}) and (y_{n_k}) converge to $x \in K$. Because f is assumed to be continuous at x, we have $\lim f(x_{n_k}) = f(x)$ and $\lim f(y_{n_k}) = f(x)$, which implies

$$\lim(f(x_{n_k}) - f(y_{n_k})) = 0.$$

A contradiction arises when we recall that (x_n) and (y_n) were chosen to satisfy

$$|f(x_n) - f(y_n)| \geq \epsilon_0$$

for all $n \in \mathbf{N}$. We conclude, then, that f is indeed uniformly continuous on K. $\square$

Example 4. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

On $[1,\infty)$ we have $x, y \geq 1$

$$\Rightarrow |\sqrt{x} - \sqrt{y}| = \left| \frac{x-y}{\sqrt{x}+\sqrt{y}} \right| \leq |x-y| \frac{1}{2}$$

So given $\varepsilon > 0$, choose $\delta = 2\varepsilon$ and then $|f(x) - f(y)| < (2\varepsilon)\frac{1}{2} = \varepsilon$ when $|x-y| < \delta$

$\Rightarrow f$ is uniformly continuous on $[1,\infty)$

$[0,1]$ is compact $\Rightarrow f$ is uniformly continuous on $[0,1]$

$\Rightarrow f$ is uniformly continuous on $[0,\infty)$

Example 5 (Lipschitz Functions)**.** A function $f : A \to \mathbf{R}$ is called <u>Lipschitz</u> if there exists a bound $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x \neq y \in A$. Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.

(a) Show that if $f : A \to \mathbf{R}$ is Lipschitz, then it is uniformly continuous on A.

(b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

a) f Lipschitz $\Rightarrow |f(x) - f(y)| \leq M|x-y| \ \forall x, y \in A$

Given $\varepsilon > 0$ choose $\delta = \varepsilon / M$. Then $|x-y| < \delta$ implies

$$|f(x) - f(y)| < M \frac{\varepsilon}{M} = \varepsilon$$

$\Rightarrow f$ is uniformly continuous on A.

b) Consider $f(x) = \sqrt{x}$, which is uniformly continuous on $[0,1]$.

If $y = 0$ and $x > 0$, then

$$\left| \frac{f(x) - f(y)}{x-y} \right| = \left| \frac{\sqrt{x}}{x} \right| = \frac{1}{\sqrt{x}},$$ which is not bounded near 0.

So no.

4.5 The Intermediate Value Theorem

Theorem 4.5.1 (Intermediate Value Theorem). *Let $f : [a, b] \to \mathbf{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$ where $f(c) = L$.*

Theorem 4.5.2 (Preservation of Connected Sets). *Let $f : G \to \mathbf{R}$ be continuous. If $E \subseteq G$ is connected, then $f(E)$ is connected as well.*

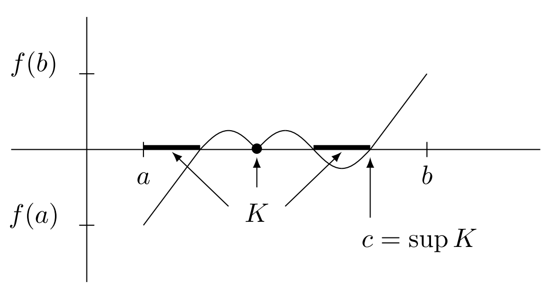
Proof. Let $f(E) = A \cup B$ where A and B are disjoint and nonempty and let

$$C = \{x \in E : f(x) \in A\} \quad \text{and} \quad D = \{x \in E : f(x) \in B\}.$$

The sets C and D are called the preimages of A and B, respectively. Using the properties of A and B, it is straightforward to check that C and D are nonempty and disjoint and satisfy $E = C \cup D$. Now, we are assuming E is a connected set, so by Theorem 3.4.2, there exists a sequence (x_n) contained in one of C or D with $x = \lim x_n$ contained in the other. Finally, because f is continuous at x, we get $f(x) = \lim f(x_n)$. Thus, it follows that $f(x_n)$ is a convergent sequence contained in either A or B while the limit $f(x)$ is an element of the other. Applying Theorem 3.4.2 again, the proof is complete. □

Proof of Theorem 4.5.1. **I.** (*First approach using AoC.*) Consider the special case where f is a continuous function satisfying $f(a) < 0 < f(b)$ and show that $f(c) = 0$ for some $c \in (a, b)$. First let

$$K = \{x \in [a, b] : f(x) \leq 0\}.$$



Notice that K is bounded above by b, and $a \in K$ so K is not empty. Thus we may appeal to the Axiom of Completeness to assert that $c = \sup K$ exists.

There are three cases to consider:

$$f(c) > 0, \; f(c) < 0, \text{ and } f(c) = 0.$$

Assume, for contradiction, that $f(c) > 0$. If we set $\epsilon_0 = f(c)$, then the continuity of f implies that there exists a $\delta_0 > 0$ with the property that

$x \in V_{\delta_0}(c)$ implies $f(x) \in V_{\epsilon_0}(f(c))$. But this implies that $f(x) > 0$ and thus $x \notin K$ for all $x \in V_{\delta_0}(c)$. What this means is that if c is an upper bound on K, then $c - \delta_0$ is a smaller upper bound, violating the definition of the supremum. We conclude that $f(x) > 0$ is not allowed.

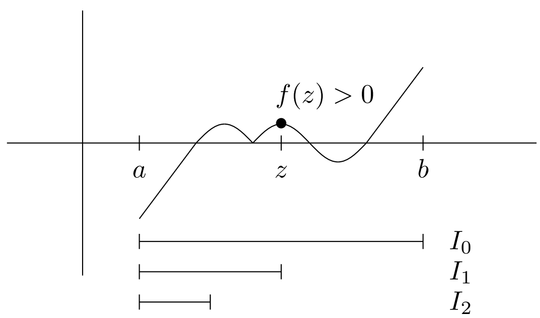
Now assume that $f(c) < 0$. This time, the continuity of f allows us to produce a neighborhood $V_{\delta_1}(c)$ where $x \in V_{\delta_1}(c)$ implies $f(x) < 0$. But this implies that a point such as $c + \delta_1/2$ is an element of K, violating the fact that c is an upper bound for K. It follows that $f(c) < 0$ is also impossible, and we conclude that $f(c) = 0$ as desired.

This proves the theorem for the special case where $L = 0$. To prove the more general version, we consider the auxiliary function $h(x) = f(x) - L$ which is certainly continuous. From the special case just considered we know $h(c) = 0$ for some point $c \in (a, b)$ from which it follows that $f(c) = L$.

II. (*Second approach using NIP.*) Again, consider the special case where $L = 0$ and $f(a) < 0 < f(b)$. Let $I_0 = [a, b]$, and consider the midpoint

$$z = (a + b)/2.$$

If $f(z) \geq 0$, then set $a_1 = a$ and $b_1 = z$. If $f(z) < 0$, then set $a_1 = z$ and $b_1 = b$. In either case, the interval $I_1 = [a_1, b_1]$ has the property that f is negative at the left endpoint and nonnegative at the right.



By repeating this construction, we get a nested sequence of intervals $I_n = [a_n, b_n]$ where $f(a_n) < 0$ and $f(b_n) \geq 0$ for all $n \in \mathbf{N}$. By the Nested Interval Property, there exists a point $c \in \bigcap_{n=1}^{\infty} I_n$. The fact that the lengths of the intervals are tending to zero means that the two sequences (a_n) and (b_n) each converge to c.

Because f is continuous at c, we get $f(c) = \lim f(a_n)$ where $f(a_n) < 0$ for all n. Then the Order Limit Theorem implies $f(c) \leq 0$. Because we also have $f(c) = \lim f(b_n)$ with $f(b_n) \geq 0$, it must be that $f(c) \geq 0$. We conclude that $f(c) = 0$. $\square$

Example 1. Show how the Intermediate Value Theorem follows as a corollary to Theorem 4.5.2.

$[a,b]$ is connected

$\Rightarrow f([a,b])$ is connected

$f(a) \in f([a,b])$ and $f(b) \in f([a,b])$

$\Rightarrow L \in f([a,b])$

$\Rightarrow \exists\, c \in (a,b)$ s.t. $L = f(c)$

Definition 4.5.1. A function f has the intermediate value property on an interval $[a,b]$ if for all $x < y$ in $[a,b]$ and all L between $f(x)$ and $f(y)$, it is always possible to find a point $c \in (x,y)$ where $f(c) = L$.

Example 2. A function f is increasing on A if $f(x) \leq f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a,b]$ and satisfies the intermediate value property, then f is continuous on $[a,b]$.

Fix $c \in (a,b)$ and let $\varepsilon > 0$.

f is increasing $\Rightarrow f(a) \leq f(c)$. If $f(c) - \varepsilon/2 < f(a)$, then set $x_1 = a$.

If $f(a) \leq f(c) - \varepsilon/2$, then by IVP $\exists\, x_1 < c$ where $f(x_1) = f(c) - \varepsilon/2$.

Either way, $x \in (x_1, c] \Rightarrow f(c) - \varepsilon/2 = f(x_1) \leq f(x) \leq f(c)$

Similarly, $\exists\, x_2 > c$ s.t. $f(c) \leq f(x) \leq f(x_2) = f(c) + \varepsilon/2$ when $x \in [c, x_2)$

Set $\delta = \min\{c - x_1, x_2 - c\}$.

Then $f(c) - \varepsilon/2 \leq f(x) \leq f(c) + \varepsilon/2$ when $|x - c| < \delta$.

The case where c is an endpoint is similar.

4.6 Sets of Discontinuity

Remark 1. Given a function $f : \mathbf{R} \to \mathbf{R}$, define $D_f \subseteq \mathbf{R}$ to be the set of points where the function f fails to be continuous. Dirichlet's function $g(x)$ has $D_g = \mathbf{R}$. The modification $h(x)$ of Dirichlet's function has $D_h = \mathbf{R} \setminus \{0\}$, zero being the only point of continuity. Finally, for Thomae's function $t(x)$, $D_t = \mathbf{Q}$.

Example 1. Using modifications of these functions, construct a function $f : \mathbf{R} \to \mathbf{R}$ so that

(a) $D_f = \mathbf{Z}^c$.

(b) $D_f = \{x : 0 < x \leq 1\}$.

Let g be the Dirichlet function.

a) $f(x) = g(x)\sin \pi x$ is continuous at each integer and discontinuous everywhere else.

b) For $x \in [0,1]$ set $f(x) = xg(x)$ and $f(x) = 0$ otherwise.

Example 2. Given a countable set $A = \{a_1, a_2, a_3, \ldots\}$, define $f(a_n) = 1/n$ and $f(x) = 0$ for all $x \notin A$. Find D_f.

If $x \in A$, then since A is countable A^c is dense, so for any $\delta > 0$ we can find $y \in V_\delta(x)$ s.t. $y \notin A$, i.e. $f(y) = 0 \Rightarrow |f(x) - f(y)| = \frac{1}{n}$ $\forall \delta > 0 \Rightarrow f$ is not continuous at $x \in A$.

If $x \notin A$, then every sequence $(x_n) \to x$ has $f(x_n) \to 0 = f(x)$, since $1/n \to 0$ for $x_n \in A$ and $x_n \notin A$ are all $0 \Rightarrow f$ is continuous at $x \notin A$.

$\Rightarrow D_f = A$ (generalization of Thomae's function, set $A = \mathbb{Q}$)

Definition 4.6.1. A function $f : A \to \mathbf{R}$ is increasing on A if $f(x) \leq f(y)$ whenever $x < y$ and decreasing if $f(x) \geq f(y)$ whenever $x < y$ in A. A monotone function is one that is either increasing or decreasing.

Definition 4.6.2. Given a limit point c of a set A and a function $f : A \to \mathbf{R}$, we write

$$\lim_{x \to c^+} f(x) = L$$

if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta$.

Equivalently, in terms of sequences, $\lim_{x \to c^+} f(x) = L$ if $\lim f(x_n) = L$ for all sequences (x_n) satisfying $x_N > c$ and $\lim(x_n) = c$.

Example 3. State a similar definition for the left-hand limit

$$\lim_{x\to c^-} f(x) = L.$$

$\lim_{x\to c^-} f(x) = L$ if for all $\varepsilon>0$ there exists a $\delta>0$ s.t. $|f(x)-L|<\varepsilon$ whenever $0<x-c<\delta$.

Theorem 4.6.1. *Given $f : A \to \mathbf{R}$ and a limit point c of A, $\lim_{x\to c} f(x) = L$ if and only if*

$$\lim_{x\to c^-} f(x) = L \text{ and } \lim_{x\to c^+} f(x) = L.$$

Example 4. Prove Theorem 4.6.1.

($\Rightarrow$) $\lim_{x\to c} f(x)=L \Rightarrow$ given $\varepsilon>0$, $\exists\,\delta>0$ s.t. $|f(x)-L|<\varepsilon$ when $0<|x-c|<\delta$.
the same δ shows $\lim_{x\to c^-} f(x)=L$ and $\lim_{x\to c^+} f(x)=L$.

($\Leftarrow$) Given $\varepsilon>0$, $\exists\,\delta_1>0$ s.t. $|f(x)-L|<\varepsilon$ when $0<x-c<\delta_1$ and $\exists\,\delta_2>0$ s.t. $|f(x)-L|<\varepsilon$ when $0<c-x<\delta_2$. Set $\delta=\min\{\delta_1,\delta_2\}$ and then $|f(x)-L|<\varepsilon$ for all $0<|x-c|<\delta \Rightarrow \lim_{x\to c} f(x)=L$

Remark 2. Generally speaking, discontinuities can be divided into three categories:

(i) If $\lim_{x\to c} f(x)$ exists but has a value different from $f(c)$, the discontinuity at c is called removable.

(ii) If $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$, then f has a jump discontinuity at c.

(iii) If $\lim_{x\to c} f(x)$ does not exist for some other reason, then the discontinuity at c is called an essential discontinuity.

Example 5. Prove that the only type of discontinuity a monotone function can have is a jump discontinuity.

First consider f increasing and set $A=\{f(x) : x<c\}$ for some $c\in\mathbb{R}$. Then A is bounded by $f(c)$. By AoC, we can set $L=\sup A$. Let $\varepsilon>0$. Then L is an upper bound, so $\exists\, x_0<c$ s.t. $L-\varepsilon<f(x_0)\le L$. Since f is increasing, choosing $\delta=c-x_0$ implies $L-\varepsilon<f(x_0)\le f(x)\le L$ when $0<c-x<\delta \Rightarrow \lim_{x\to c^-} f(x)=L$. Similarly, $\lim_{x\to c^+} f(x)=L'$ where $L'=\inf\{f(x) : x>c\}$.
$\Rightarrow L\le f(c)\le L'$
If $L=L'$, then f is continuous at c. If $L<L'$, then f has a jump discontinuity at c.
The case where f is decreasing is similar.

Example 6. Construct a bijection between the set of jump discontinuities of a monotone function f and a subset of $\mathbf{Q}$. Conclude that D_f for a monotone function f must either be finite or countable, but not uncountable.

Let f be increasing and c be a point of discontinuity. Set $\lim_{x \to c^-} f(x) = L_c$ and $\lim_{x \to c^+} f(x) = L'_c$.
Then $L_c < L'_c$. $\mathbb{Q}$ is dense in $\mathbb{R} \Rightarrow \exists\, r_c \in \mathbb{Q}$ s.t. $L_c < r_c < L'_c$.
Also, $c_1 < c_2 \Rightarrow r_{c_1} < r_{c_2}$, so $\phi(c) = r_c$ is 1-1. $\Rightarrow D_f$ is finite or countable.

Definition 4.6.3. A set that can be written as the countable union of closed sets is in the class F_σ.

Example 7. (a) Show that in Dirichlet's function, the modified Dirichlet function, and Thomae's function we get an F_σ set as the set where the function is discontinuous.

(b) Show that the two sets of discontinuity in Example 1 are F_σ sets.

a) For Dirichlet's function, $\mathbb{R}$ is closed. For the modified Dirichlet function, set $A_n = (-\infty, 1/n] \cup [1/n, \infty)$, which is closed. Then $\mathbb{R} \setminus \{0\} = \bigcup_{n=1}^{\infty} A_n$ is F_σ.
For Thomae's function, $\mathbb{Q}$ is the countable union of singleton sets, which are closed.

b) Let $n \in \mathbb{N}$ satisfy $n \geq 3$ and let $z \in \mathbb{Z}$. Define $U(n,z) = [z + (1/n), (z+1) - (1/n)]$.
Then $U(n) = \bigcup_{z \in \mathbb{Z}} (n,z)$ is closed and $\mathbb{Z}^c = \bigcup_{n=3}^{\infty} U(n)$.
For the second example write $(0,1] = \bigcup_{n=1}^{\infty} [1/n, 1]$.

Definition 4.6.4. Let f be defined on $\mathbf{R}$, and let $\alpha > 0$. The function f is <u>α-continuous at $x \in \mathbf{R}$</u> if there exists a $\delta > 0$ such that for all $y, z \in (x - \delta, x + \delta)$ it follows that $|f(y) - f(z)| < \alpha$.

Remark 3. Given a function f on $\mathbf{R}$, define D_f^α to be the set of points where the function f fails to be α-continuous. In other words,

$$D_f^\alpha = \{x \in \mathbf{R} : f \text{ is not } \alpha\text{-continuous at } x\}.$$

Example 8. Prove that, for a fixed $\alpha > 0$, the set D_f^α is closed.

Let c be a limit point of D_α and let $\delta > 0$. $\Rightarrow \exists\, x' \in D_\alpha$ s.t. $x' \in V_{\delta/2}(c)$
$\Rightarrow \exists\, y, z \in V_{\delta/2}(x')$ s.t. $|f(y) - f(z)| \geq \alpha$.
$V_{\delta/2}(x') \subseteq V_\delta(c) \Rightarrow \forall\, \delta > 0\ \exists\, y, z \in V_\delta(c)$ s.t. $|f(y) - f(z)| \geq \alpha \Rightarrow c \in D_\alpha$

Theorem 4.6.2. *Let $f : \mathbf{R} \to \mathbf{R}$ be an arbitrary function. Then D_f is an F_σ set.*

Proof. If $\alpha < \alpha'$ and $c \in D_f^{\alpha'}$, then given $\delta > 0$, there exist $y, z \in V_\delta(c)$ satisfying

$$|f(y) - f(z)| \geq \alpha' > \alpha.$$

Thus $c \in D_f^\alpha$ as well, i.e., $D_f^{\alpha'} \subseteq D_f^\alpha$.

Now suppose f is continuous at x. Then given fixed $\alpha > 0$, we know there exists a $\delta > 0$ such that

$$|f(y) - f(x)| < \frac{\alpha}{2} \quad \text{provided } y \in V_\delta(x).$$

Thus, if $y, z \in V_\delta(x)$ we then get

$$\begin{aligned} |f(y) - f(z)| &\leq |f(y) - f(x)| + |f(x) - f(z)| \\ &< \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha, \end{aligned}$$

and we conclude that f is α-continuous at x. The contrapositive of this conclusion is that if f is α-continuous at x, then it certainly cannot be continuous at x. That is, $D_f^\alpha \subseteq D_f$.

Now assume f is not continuous at x. Negating the ϵ-δ definition of continuity we get that there exists an $\epsilon_0 > 0$ with the property that for all $\delta > 0$ there exists a point $y \in V_\delta(x)$ where $|f(y) - f(x)| \geq \epsilon_0$. Noting simply that both $x, y \in V_\delta(x)$, we conclude that f is not α-continuous for $\alpha = \epsilon_0$ (or anything smaller.)

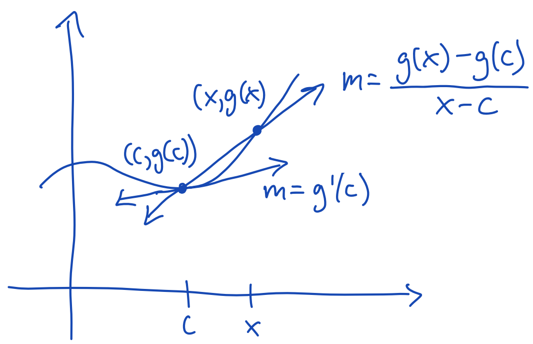
To prove $D_f = \bigcup_{n=1}^\infty D_f^{1/n}$ we argue for inclusion each way. If $x \in D_f$, then we have just shown that $x \in D_f^{\epsilon_0}$ for some $\epsilon_0 > 0$. Choosing $n_0 \in \mathbf{N}$ small enough so that $1/n_0 \leq \epsilon_0$, it follows that $x \in D_f^{1/n_0}$. This proves $D_f \subseteq \bigcup_{n=1}^\infty D_f^{1/n}$.

For the reverse inclusion we observe that we already showed $D_f^{\alpha'} \subseteq D_f^\alpha$ when $\alpha < \alpha'$, so $D_f^{1/n} \subseteq D_f$ for all $n \in \mathbf{N}$. Because each $D_f^{1/n}$ is closed, the result follows. $\square$

Chapter 5

The Derivative

5.1 Discussion: Are Derivatives Continuous?



$$g'(c) = \lim_{x\to c} \frac{g(x)-g(c)}{x-c}$$

$$g_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$g_0(x)$ is not continuous at $x=0$, but $g_1(x)$ is

$g_1'(0) = \lim_{x\to 0} \frac{g_1(x)}{x} = \lim_{x\to 0} \sin(1/x)$ DNE

$g_2'(0) = \lim_{x\to 0} x\sin(1/x) = 0$

$$g_2'(x) = \begin{cases} -\cos(1/x) + 2x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

but $\lim_{x\to 0} g_2'(x)$ DNE, so g_2' is not continuous

g_2' has an essential discontinuity

Does there exist a derivative with a jump discontinuity?

i.e., does there exist h s.t. $h'(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$?

5.2 Derivatives and the Intermediate Value Property

Definition 5.2.1 (Differentiability)**.** Let $g : A \to \mathbf{R}$ be a function defined on an interval A. Given $c \in A$, the derivative of G at c is defined by

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c},$$

provided this limit exists. In this case we say g is differentiable at c. If g' exists for all points $c \in A$, we say that g is differentiable on A.

Example 1. (i) Calculate the derivative of $f(x) = x^n$ where $n \in \mathbf{N}$ at an arbitrary point c in $\mathbf{R}$.

(ii) Show that $g(x) = |x|$ is not differentiable at zero.

(i) $x^n - c^n = (x-c)(x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1})$

$\Rightarrow f'(c) = \lim_{x \to c} \frac{x^n - c^n}{x - c}$

$= \lim_{x \to c} (x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1})$

$= c^{n-1} + c^{n-1} + \cdots + c^{n-1} = nc^{n-1}$

(ii) $\lim_{x \to 0^+} \frac{|x|}{x} = 1$ and $\lim_{x \to 0^-} \frac{|x|}{x} = -1 \Rightarrow g'(0) = \lim_{x \to 0} \frac{|x|}{x}$ DNE

Theorem 5.2.1. *If $g : A \to \mathbf{R}$ is differentiable at a point $c \in A$, then g is continuous at c as well.*

Proof. We are assuming that

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

exists, and we want to prove that $\lim_{x \to c} g(x) = g(c)$. But notice that the Algebraic Limit Theorem for functional limits allows us to write

$$\lim_{x \to c}(g(x) - g(c)) = \lim_{x \to c} \left(\frac{g(x) - g(c)}{x - c} \right) (x - c) = g'(c) \cdot 0 = 0.$$

It follows that $\lim_{x \to c} g(x) = g(c)$. $\square$

Theorem 5.2.2 (Algebraic Differentiability Theorem)**.** *Let f and G be functions defined on an interval A, and assume both are differentiable at some point $c \in A$. Then,*

(i) $(f+g)'(c) = f'(c) + g'(c)$,

(ii) $(kf)'(c) = kf'(c)$, *for all* $k \in \mathbf{R}$,

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$, *and*

(iv) $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$, *provided that* $g(c) \neq 0$.

Proof. Statements (i) and (ii) are left as exercises. To prove (iii), we rewrite the difference quotient as

$$\begin{aligned}\frac{(fg)(x) - (fg)(c)}{x - c} &= \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x - c} \\ &= f(x)\left[\frac{g(x) - g(c)}{x - c}\right] + g(c)\left[\frac{f(x) - f(c)}{x - c}\right].\end{aligned}$$

Because f is differentiable at c, it is continuous there and thus $\lim_{x \to c} f(x) = f(c)$. This fact, together with the functional-limit version of the Algebraic Limit Theorem (Corollary 4.2.1), justifies the conclusion

$$\lim_{x \to c} \frac{(fg)(x) - (fg)(c)}{x - c} = f(c)g'(c) + f'(c)g(c).$$

A similar proof of (iv) is possible, or we can use an argument based on the next result. □

Theorem 5.2.3 (Chain Rule)**.** *Let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$ satisfy $f(A) \subseteq B$ so that the composition $g \circ f$ is defined. If f is differentiable at $c \in A$ and if g is differentiable at $f(c) \in B$, then $g \circ f$ is differentiable at c with $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.*

Proof. Because g is differentiable at c, we know that

$$g'(f(c)) = \lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)}.$$

Another way to assert this same fact is to let $d(y)$ be the difference quotient

$$d(y) = \frac{g(y) - g(f(c))}{y - f(c)}, \tag{1}$$

and observe that $\lim_{y \to f(c)} d(y) = g'(f(c))$. At the moment, $d(y)$ is not defined when $y = f(c)$, but it should seem natural to declare that $d(f(c)) = g'(f(c))$,

so that d is continuous at $f(c)$.
Equation (1) can be rewritten as

$$g(y) - g(f(c)) = d(y)(y - f(c)). \tag{2}$$

Observe that this equation holds *for all* $y \in B$ *including* $y = f(c)$. Thus, we are free to substitute $y = f(t)$ for any arbitrary $t \in A$. If $t \neq c$, we can divide equation (2) by $(t - c)$ to get

$$\frac{g(f(t)) - g(f(c))}{t - c} = d(f(t))\frac{(f(t) - f(c))}{t - c}$$

for all $t \neq c$. Finally, taking the limit as $t \to c$ and applying the Algebraic Limit Theorem together with Theorem 4.3.3 yields the desired formula. □

Example 2. (a) Use Definition 5.2.1 to produce the proper formula for the derivative of $h(x) = 1/x$.

(b) Combine the result in part (a) with the Chain Rule (Theorem 5.2.3) to supply a proof for part (iv) of Theorem 5.2.2.

(c) Supply a direct proof of Theorem 5.2.2 by algebraically manipulating the difference quotient for (f/g) in a style similar to the proof of Theorem 5.2.2 (iii).

(a) For $c \neq 0$, $h'(c) = \lim_{x \to c} \frac{1/x - 1/c}{x - c} = \lim_{x \to c} \frac{(c-x)/xc}{x-c} = \lim_{x \to c} \frac{-1}{xc} = \frac{-1}{c^2}$

(b) $\left(\frac{1}{g(x)}\right)' = (h \circ g)'(x) = \frac{-g'(x)}{[g(x)]^2}$

$$\left(\frac{f}{g}\right)'(x) = [f(x)(h\circ g)(x)]' = f'(x)(h\circ g)(x) + f(x)(h\circ g)'(x)$$

$$= \frac{f'(x)}{g'(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$g(c) \neq 0$

(c) $$\frac{(f/g)(x) - (f/g)(c)}{x - c} = \frac{1}{x-c}\left(\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}\right) = \frac{1}{x-c}\left(\frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)}\right)$$

$$= \frac{1}{x-c}\left(\frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)}{g(x)g(c)}\right)$$

$$= \frac{1}{g(x)g(c)}\left(g(c)\frac{f(x)-f(c)}{x-c} - f(c)\frac{g(x)-g(c)}{x-c}\right)$$

$$\Rightarrow \left(\frac{f}{g}\right)'(c) = \frac{1}{[g(c)]^2}\left(g(c)f'(c) - f(c)g'(c)\right)$$

Example 3. Given a differentiable function $f : A \to \mathbf{R}$, let's say that f is uniformly differentiable on A if, given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\frac{f(x)-f(y)}{x-y} - f'(y)\right| < \epsilon \quad \text{whenever } 0 < |x-y| < \delta.$$

(a) Is $f(x) = x^2$ uniformly differentiable on $\mathbf{R}$? How about $g(x) = x^3$?

(b) Show that if a function is uniformly differentiable on an interval A, then the derivative must be continuous on A.

(c) Is there a theorem analogous to Theorem 4.4.4 for differentiation? Are functions that are differentiable on a closed interval $[a, b]$ necessarily uniformly differentiable?

(a) $\left|\frac{x^2-y^2}{x-y} - 2y\right| = \left|\frac{(x-y)(x+y)}{x-y} - 2y\right| = |x-y|$

Given $\varepsilon>0$, choose $\delta=\varepsilon$. Then $|x-y|<\delta=\varepsilon$ implies $\left|\frac{x^2-y^2}{x-y} - 2y\right| = |x-y| < \varepsilon$

$\Rightarrow f(x)=x^2$ is uniformly differentiable on $\mathbb{R}$.

$\left|\frac{x^3-y^3}{x-y} - 3y^2\right| = \left|\frac{(x-y)(x^2+xy+y^2)}{x-y} - 3y^2\right| = |x^2+xy-2y^2| = |x+2y|\,|x-y|$

Let $\delta>0$ and let $x=\frac{2+\delta^2}{3\delta}$, $y=\frac{4-\delta^2}{6\delta}$.

Then $|x-y| = \frac{\delta}{2} < \delta$ but $\left|\frac{x^3-y^3}{x-y} - 3y^2\right| = |x+2y|\,|x-y| = \frac{\delta}{2}\left(\frac{2+\delta^2}{3\delta} + \frac{4-\delta^2}{3\delta}\right) = 1$

$\Rightarrow f(x)=x^3$ is not uniformly differentiable on $\mathbb{R}$.

(b) Let $\delta>0$ be s.t. $\left|\frac{f(x)-f(y)}{x-y} - f'(y)\right| < \frac{\varepsilon}{2}$ when $0<|x-y|<\delta$.

Let $c\in A$ and $|x-c|<\delta$. Then $\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \frac{\varepsilon}{2}$ and $\left|\frac{f(c)-f(x)}{c-x} - f'(x)\right| < \frac{\varepsilon}{2}$

$$\Rightarrow |f'(x)-f'(c)| = \left|f'(x) - \frac{f(c)-f(x)}{c-x} + \frac{f(x)-f(c)}{x-c} - f'(c)\right|$$

$$\leq \left|\frac{f(c)-f(x)}{c-x} - f'(x)\right| + \left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow f'(x)$ is continuous at $c\in A$.

(C) Let $g_2(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ and set $t_n = 1/(2n\pi)$ and $x_n = 0$.

Then $|x_n - t_n| \to 0$ while

$$\left| \frac{g_2(x_n) - g_2(t_n)}{x_n - t_n} - g_2'(t_n) \right| = |t_n \sin(1/t_n) + \cos(1/t_n) - 2t_n \sin(1/t_n)|$$

$$= |\cos(1/t_n) - t_n \sin(1/t_n)| = 1 \quad \forall n \in \mathbb{N}$$

$\Rightarrow g_2(x)$ is not uniformly differentiable, so no.

Theorem 5.2.4 (Interior Extremum Theorem)**.** *Let f be differentiable on an open interval (a, b). If f attains a maximum value at some point $c \in (a, b)$ (i.e., $f(c) \geq f(x)$ for all $x \in (a, b)$), then $f'(c) = 0$. The same is true if $f(c)$ is a minimum value.*

Proof. Because c is in the open interval (a, b), we can construct two sequences (x_n) and (y_n), which converge to c and satisfy $x_n < c < y_n$ for all $n \in \mathbf{N}$. The fact that $f(c)$ is a maximum implies that $f(y_n) - f(c) \geq 0$ for all n, and thus

$$f'(c) = \lim_{n\to\infty} \frac{f(y_n) - f(c)}{y_n - c} \leq 0$$

by the Order Limit Theorem (Theorem 2.3.3). In a similar way,

$$\frac{f(x_n) - f(c)}{x_n - c} \geq 0$$

for each x_n because both numerator and denominator are negative. This implies that

$$f'(c) = \lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} \geq 0,$$

and therefore $f'(c) = 0$, as desired. □

Theorem 5.2.5 (Darboux's Theorem)**.** *If f is differentiable on an interval $[a,b]$, and if α satisfies $f'(a) < \alpha < f'(b)$ (or $f'(a) > \alpha > f'(b)$), then there exists a point $c \in (a,b)$ where $f'(c) = \alpha$.*

Proof. We first simplify matters by defining a new function $g(x) = f(x) - \alpha x$ on $[a,b]$. Notice that g is differentiable on $[a,b]$ with $g'(x) = f'(x) - \alpha$. In terms of g, our hypothesis states that $g'(a) < 0 < g'(b)$, and we hope to show that $g'(c) = 0$ for some $c \in (a,b)$.

We start by proving that there exists $x \in (a,b)$ where $g(x) < g(a)$. Let (x_n) be a sequence in (a,b) satisfying $(x_n) \to a$. Then we have

$$g'(a) = \lim_{n\to\infty} \frac{g(x_n) - g(a)}{x_n - a} < 0.$$

The denominator is always positive. If the numerator were always positive then the Order Limit Theorem would imply $g'(a) \geq 0$. Because we know this is not the case, we may conclude that the numerator is eventually negative and thus $g(x) < g(a)$ for some x near a. The proof that there exists $y \in (a,b)$ where $g(y) < g(b)$ is similar.

We must now show that $g'(c) = 0$ for some $c \in (a,b)$. Because g is differentiable on the compact set $[a,b]$ it must also be continuous here, and so by the Extreme Value Theorem (Theorem 4.4.2), g attains a minimum at a point $c \in [a,b]$. From our work in (a) we know that the minimum of g is neither $g(a)$ nor $g(b)$, and therefore $c \in (a,b)$. Finally, the Interior Extremum Theorem (Theorem 5.2.4) allows us to conclude $g'(c) = 0$.

To prove the general result stated in the theorem we just observe that $g'(c) = 0$ is equivalent to the conclusion $f'(c) = \alpha$. □

5.3 The Mean Value Theorems

Theorem 5.3.1 (Rolle's Theorem)**.** *Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ where $f'(c) = 0$.*

Proof. Because f is continuous on a compact set, f attains a maximum and a minimum. If both the maximum and minimum occur at the endpoints, then f is necessarily a constant function and $f'(x) = 0$ on all of (a, b). In this case, we can choose c to be any point we like. On the other hand, if either the maximum or minimum occurs at some point c in the interior (a, b), then it follows from the Interior Extremum Theorem (Theorem 5.2.4) that $f'(c) = 0$. □

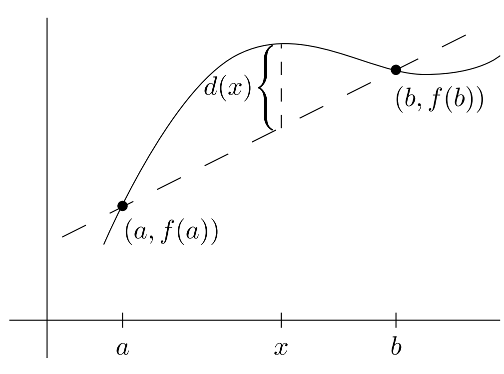
Theorem 5.3.2 (Mean Value Theorem)**.** *If $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ where*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Notice that the Mean Value Theorem reduces to Rolle's Theorem in the case where $f(a) = f(b)$. The strategy of the proof is to reduce the more general statement to this special case.

The equation of the line through $(a, f(a))$ and $(b, f(b))$ is

$$y = \left(\frac{f(b) - f(a)}{b - a}\right)(x - a) + f(a).$$



We want to consider the difference between this line and the function $f(x)$. To this end, let

$$d(x) = f(x) - \left[\left(\frac{f(b) - f(a)}{b - a}\right)(x - a) + f(a)\right],$$

and observe that d is continuous on $[a, b]$, differentiable on (a, b), and satisfies $d(a) = 0 = d(b)$. Thus, by Rolle's Theorem, there exists a point $c \in (a, b)$ where $d'(c) = 0$. Because

$$d'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we get

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a},$$

which completes the proof. $\square$

Example 1. Let h be a differentiable function defined on the interval $[0, 3]$, and assume that $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

(a) Argue that there exists a point $d \in [0, 3]$ where $h(d) = d$.

(b) Argue that at some point c we have $h'(c) = 1/3$.

(c) Argue that $h'(x) = 1/4$ at some point in the domain.

(a) Set $g(x) = x - h(x)$. $g(1) = -1$ and $g(3) = 1 \Rightarrow \exists\, d \in [0,3]$ where $g(d) = 0$ by IVT.
$\Rightarrow h(d) = d$

(b) By MVT $\exists\, c \in (0,3)$ s.t. $h'(c) = \dfrac{h(3) - h(0)}{3 - 0} = \dfrac{2-1}{3} = \dfrac{1}{3}$

(c) By Rolle's Theorem $\exists\, a' \in (1,3)$ where $h'(a) = 0$. By (b) $\exists\, c$ s.t. $h'(c) = \frac{1}{3}$
$0 < \frac{1}{4} < \frac{1}{3} \Rightarrow h'(x) = 1/4$ at some point between c and a by Darboux's Theorem

Example 2. A <u>fixed point</u> of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Assume f has fixed points x_1 and x_2 with $x_1 \neq x_2$.
Then $f(x_1) = x_1$ and $f(x_2) = x_2$, so by MVT $\exists\, c$ s.t.

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{x_1 - x_2}{x_1 - x_2} = 1,$$ a contradiction.

Corollary 5.3.1. *If $g : A \to \mathbf{R}$ is differentiable on an interval A and satisfies $g'(x) = 0$ for all $x \in A$, then $g(x) = k$ for some constant $k \in \mathbf{R}$.*

Proof. Take $x, y \in A$ and assume $x < y$. Applying the Mean Value Theorem to g on the interval $[x, y]$, we see that

$$g'(c) = \frac{g(y) - g(x)}{y - x}$$

for some $c \in A$. Now, $g'(c) = 0$, so we conclude that $g(y) = g(x)$. Set k equal to this common value. Because x and y are arbitrary, it follows that $g(x) = k$ for all $x \in A$. □

Corollary 5.3.2. *If f and g are differentiable functions on an interval A and satisfy $f'(x) = g'(x)$ for all $x \in A$, then $f(x) = g(x) + k$ for some constant $k \in \mathbf{R}$.*

Proof. Let $h(x) = f(x) - g(x0$ and apply Corollary 5.3.1 to the differentiable function h. □

Theorem 5.3.3 (Generalized Mean Value Theorem)**.** *If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ where*

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If g' is never zero on (a, b), then the conclusion can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Example 3. Prove Theorem 5.3.3.

Let $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$.

Then h is continuous on $[a,b]$ and differentiable on (a,b). Also, $h(a) = g(a)f(b) - f(a)g(b)$.

$\Rightarrow \exists\, c \in (a,b)$ where $h'(c) = 0$ by Rolle's Theorem.

So $h'(x) = [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x)$

$\Rightarrow [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0$

Theorem 5.3.4 (L'Hospital's Rule: 0/0 case)**.** *Let f and g be continuous on an interval containing a, and assume f and g are differentiable on this interval with the possible exception of the point a. If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$, then*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \quad \textit{implies} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Example 4. Prove Theorem 5.3.4.

Let $\varepsilon > 0$. $L = \lim_{x \to a} f'(x)/g'(x) \Rightarrow \exists \delta > 0$ s.t. $\left| \frac{f'(t)}{g'(t)} - L \right| < \varepsilon$ when $0 < |t - a| < \delta$.

Pick $x \in V_\delta(a)$ with $a < x$ (the case $x < a$ is similar).

By GMVT $\exists\, c \in (a, x)$ s.t. $\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}$.

$0 < |c - a| < \delta \Rightarrow \left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(c)}{g'(c)} - L \right| < \varepsilon$ when $0 < |x - a| < \delta$.

Definition 5.3.1. Given $g : A \to \mathbf{R}$ and a limit points c of A, we say that $\lim_{x \to c} g(x) = \infty$ if, for every $M > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ it follows that $g(x) \geq M$.

We can define $\lim_{x \to c} g(x) = -\infty$ in a similar way.

Theorem 5.3.5 (L'Hospital's Rule: ∞/∞ case)**.** *Assume f and g are differentiable on (a, b) and that $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim_{x \to a} g(x) = \infty$ (or $-\infty$), then*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \quad \textit{implies} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Proof. Let $\epsilon > 0$. Because $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$, there exists a $\delta_1 > 0$ such that

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \frac{\epsilon}{2}$$

for all $a < x < a + \delta_1$. For convenience of notation, let $t = a + \delta_1$ and note that t is fixed for the remainder of the argument.

Our functions are not defined at a, but for any $x \in (a, t)$ we can apply the Generalized Mean Value Theorem on the interval $[x, t]$ to get

$$\frac{f(x) - f(t)}{g(x) - g(t)} = \frac{f'(c)}{g'(c)}$$

for some $c \in (x, t)$. Our choice of t then implies

$$L - \frac{\epsilon}{2} < \frac{f(x) - f(t)}{g(x) - g(t)} < L + \frac{\epsilon}{2} \tag{1}$$

for all x in (a, t).

In an effort to isolate the fraction $\frac{f(x)}{g(x)}$, the strategy is to multiply inequality

(1) by $(g(x) - g(t))/g(x)$. We need to be sure, however, that this quantity is positive, which amounts to insisting that $1 \geq g(t)/g(x)$. Because t is fixed and $\lim_{x\to a} g(x) = \infty$, we can choose $\delta_2 > 0$ so that $g(x) \geq g(t)$ for all $a < x < a + \delta_2$. Carrying out the desired multiplication results in

$$\left(L - \frac{\epsilon}{2}\right)\left(1 - \frac{g(t)}{g(x)}\right) < \frac{f(x) - f(t)}{g(x)} < \left(L + \frac{\epsilon}{2}\right)\left(1 - \frac{g(t)}{g(x)}\right),$$

which after some algebraic manipulations yields

$$L - \frac{\epsilon}{2} + \frac{-Lg(t) + \frac{\epsilon}{2}g(t) + f(t)}{g(x)} < \frac{f(x)}{g(x)} < L + \frac{\epsilon}{2} + \frac{-Lg(t) - \frac{\epsilon}{2}g(t) + f(t)}{g(x)}.$$

Again, let's remind ourselves that t is fixed and that $\lim_{x\to a} g(x) = \infty$. Thus, we can choose a δ_3 such that $a < x < a + \delta_3$ implies that $g(x)$ is large enough to ensure that both

$$\frac{-Lg(t) + \frac{\epsilon}{2}g(t) + f(t)}{g(x)} \quad \text{and} \quad \frac{-Lg(t) - \frac{\epsilon}{2}g(t) + f(t)}{g(x)}$$

are less than $\epsilon/2$ in absolute value. Putting this all together and choosing $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ guarantees that

$$\left|\frac{f(x)}{g(x)} - L\right| < \epsilon$$

for all $a < x < a + \delta$. □

Example 5. Let $f(x) = x\sin(1/x^4)e^{-1/x^2}$ and $g(x) = e^{-1/x^2}$. Using the familiar properties of these functions, compute the limit as x approaches zero of $f(x)$, $g(x)$, $f(x)/g(x)$, and $f'(x)/g'(x)$.

$\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0} g(x) = 0$, $\lim_{x\to 0} f(x)/g(x) = 0$

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{[\sin(1/x^4) - 4(1/x^4)\cos(1/x^4)]e^{-1/x^2} + x\sin(1/x^4)e^{-1/x^2}(2/x^3)}{e^{-1/x^2}(2/x^3)}$$

$$= \lim_{x\to 0} \frac{1}{2}x^3[\sin(1/x^4) - 4(1/x^4)\cos(1/x^4)] + x\sin(1/x^4)e^{-1/x^2}$$

$$= \frac{1}{2}\lim_{x\to 0}[x^3\sin(1/x^4) - 4(1/x)\cos(1/x^4)]$$

$$= -2\lim_{x\to 0}(1/x)\cos(1/x^4) \quad \text{DNE}$$

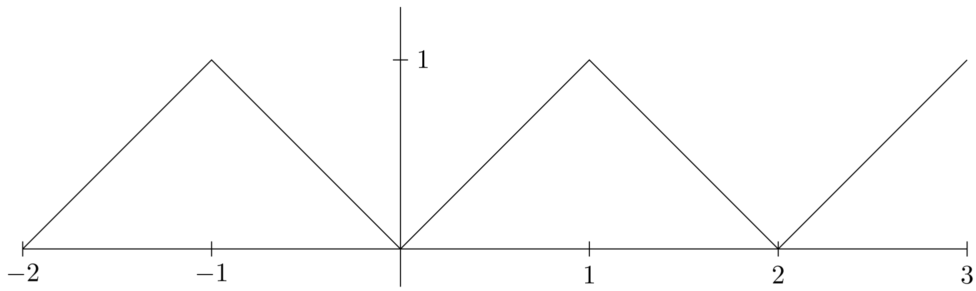
So the converse of L'Hospital does not hold.

5.4 A Continuous Nowhere-Differentiable Function

Remark 1. Define

$$h(x) = |x|$$

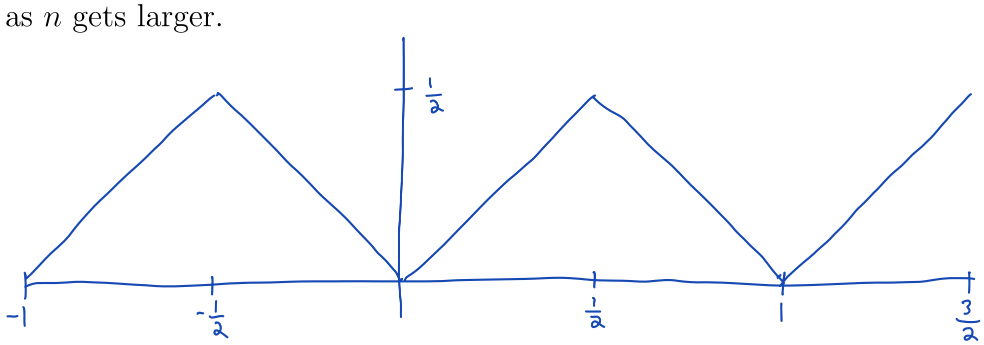
on the interval $[-1, 1]$ and extend the definition of h to all of $\mathbf{R}$ by requiring that $h(x + 2) = h(x)$. The result is a periodic "sawtooth" function.



Example 1. Sketch a graph of $(1/2)h(2x)$ on $[-2, 3]$. Give a qualitative description of the functions

$$h_n(x) = \frac{1}{2^n} h(2^n x)$$

as n gets larger.



For each n the max height is $1/2^n$ and the period is $1/2^{n-1}$.

The slopes of the line segments are ± 1 $\forall n$.

Example 2. Fix $x \in \mathbf{R}$. Argue that the series

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$$

converges absolutely and thus $g(x)$ is properly defined.

$h_n(x) \le 1 \quad \forall n$

$\Rightarrow 0 \le \frac{1}{2^n} h_n(2^n x) \le \frac{1}{2^n}$

$\sum \frac{1}{2^n}$ converges, so $g(x)$ converges by the Comparison Test.

All terms are positive, so $g(x)$ converges absolutely.

Example 3. Taking the continuity of $h(x)$ as given, reference the proper theorems from Chapter 4 that imply that the *finite* sum

$$g_m(x) = \sum_{n=0}^{m} \frac{1}{2^n} h(2^n x)$$

is continuous on $\mathbf{R}$.

For each n, $\ell(x) = 2^n x$ is continuous

$\Rightarrow h(2^n x)$ is continuous by the Composition of Continuous Functions Theorem

$\Rightarrow \frac{1}{2^n} h(2^n x)$ is continuous by the Algebraic Continuity Theorem

$\Rightarrow g_m(x) = h(x) + \frac{1}{2} h(2x) + \cdots + \frac{1}{2^m} h(2^m x)$ is continuous by the same theorem

Example 4. Consider the sequence $x_m = 1/2^m$, where $m = 0, 1, 2, \ldots$. Show that
$$\frac{g(x_m) - g(0)}{x_m - 0} = m + 1,$$
and use this to prove that $g'(0)$ does not exist.

Fix $m \in \mathbb{N}$. Then $g(x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^{n-m})$. If $n > m$ then $h(2^{n-m}) = 0$.

If $n \le m$ then $h(x) = x$, so $\frac{1}{2^n} h(2^{n-m}) = \frac{1}{2^n} 2^{n-m} = \frac{1}{2^m}$.

$$\Rightarrow g(x_m) = \sum_{n=0}^{m} \frac{1}{2^m} \Rightarrow \frac{g(x_m) - g(0)}{x_m - 0} = \frac{\sum_{n=0}^{m} 1/2^m}{1/2^m} = \sum_{n=0}^{m} 1 = m+1$$

$\Rightarrow g'(0) = \lim_{m\to\infty} \frac{g(x_m) - g(0)}{x_m - 0} = \lim_{m\to\infty}(m+1)$ DNE for $x_m = 1/2^m$ and $(x_m) \to 0$

$\Rightarrow g$ is not differentiable at 0.

Example 5. (a) Modify the previous argument to show that $g'(1)$ does not exist. Show that $g'(1/2)$ does not exist.

Let $x_m = 1/2^m$, so that $g(1+x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n(1+1/2^m)) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n + 2^{n-m})$.

If $n > m$, then $\frac{1}{2^n} h(2^n + 2^{n-m}) = 0$.

If $1 \le n \le m$, then $\frac{1}{2^n} h(2^n + 2^{n-m}) = \frac{1}{2^n} h(2^{n-m}) = \frac{1}{2^n} 2^{n-m} = \frac{1}{2^m}$

If $n = 0$, then $\frac{1}{2^n} h(2^n + 2^{n-m}) = h(1 + 1/2^m) = h(1) - 1/2^m = g(1) - 1/2^m$

$$\frac{g(1+x_m) - g(1)}{x_m} = \frac{\sum_{n=0}^{m} 1/2^n\, h(2^n + 2^{n-m}) - g(1)}{1/2^m} = \frac{\left[\sum_{n=1}^{m} 1/2^m\right] + \left(g(1) - 1/2^m\right) - g(1)}{1/2^m} = m-1$$

$\Rightarrow g$ is not differentiable at 1.

Now let $x = \frac{1}{2}$ and note that $g(\frac{1}{2}) = h(\frac{1}{2}) + \frac{1}{2} h(1)$.

Then
$$\frac{g(\frac{1}{2} + x_m) - g(\frac{1}{2})}{x_m} = \frac{\sum_{n=0}^{m} 1/2^n\, h(2^{n-1} + 2^{n-m}) - g(\frac{1}{2})}{1/2^m}$$

$$= \frac{\left[\sum_{n=2}^{m} 1/2^m\right] + h(\frac{1}{2} + \frac{1}{2^m}) + h(1 + \frac{2}{2^m}) - g(\frac{1}{2})}{1/2^m} = \frac{\left[\sum_{n=2}^{m} 1/2^m\right] + h(\frac{1}{2}) - 1/2^m + \frac{1}{2} h(1) - 1/2^m - g(\frac{1}{2})}{1/2^m} = m-3$$

$\Rightarrow g$ is not differentiable at $\frac{1}{2}$.

(b) Show that $g'(x)$ does not exist for any rational number of the form $x = p/2^k$ where $p \in \mathbf{Z}$ and $k \in \mathbf{N} \cup \{0\}$.

Let $x = \frac{p}{2^k}$ and consider $g(x+x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n(x + 1/2^m)) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(p2^{n-k} + 2^{n-m})$.

Because we will take $m \to \infty$, assume $m > k$.

If $n > m$, then $\frac{1}{2^n} h(p2^{n-k} + 2^{n-m}) = 0$.

If $k < n \leq m$, then $\frac{1}{2^n} h(p2^{n-k} + 2^{n-m}) = \frac{1}{2^n} h(2^{n-m}) = \frac{1}{2^n} 2^{n-m} = \frac{1}{2^m}$.

If $0 \leq n \leq k$, then $\frac{1}{2^n} h(p2^{n-k} + 2^{n-m}) = \frac{1}{2^n}\left[h(p2^{n-k}) \pm 2^{n-m}\right] = \frac{1}{2^n} h(2^n x) \pm 1/2^m$

$$\frac{g(x+x_m) - g(x)}{x_m} = \frac{\sum_{n=0}^{m} \frac{1}{2^n} h(p2^{n-k} + 2^{n-m}) - g(x)}{1/2^m}$$

$$= \frac{\left[\sum_{n=k+1}^{m} 1/2^m\right] + \left[\sum_{n=0}^{k} (1/2^n) h(2^n x) \pm 1/2^m\right] - g(x)}{1/2^m}$$

$$= (m-k-1) + \sum_{n=0}^{k} \pm 1 \geq m - 2k - 1$$

$\Rightarrow g$ is not differentiable at any dyadic rational point.

Example 6. (a) First prove the following general lemma: Let f be defined on an open interval J and assume f is differentiable at $a \in J$. If (a_n) and (b_n) are sequences satisfying $a_n < a < b_n$ and $\lim a_n = \lim b_n = a$, show

$$f'(a) = \lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n}.$$

(b) Now use this lemma to show that $g'(x)$ does not exist.

a) $$\lim_{n\to\infty} \frac{f(b_n)-f(a_n)}{b_n-a_n} = \lim_{n\to\infty}\left[\frac{f(b_n)-f(a)}{b_n-a_n} + \frac{f(a)-f(a_n)}{b_n-a_n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{f(b_n)-f(a)}{b_n-a}\left(\frac{b_n-a}{b_n-a_n}\right) + \frac{f(a)-f(a_n)}{a-a_n}\left(\frac{a-a_n}{b_n-a_n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{f(b_n)-f(a)}{b_n-a}\left(1-\frac{a-a_n}{b_n-a_n}\right) + \frac{f(a)-f(a_n)}{a-a_n}\left(\frac{a-a_n}{b_n-a_n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{f(b_n)-f(a)}{b_n-a} + \left(\frac{f(a)-f(a_n)}{a-a_n} - \frac{f(b_n)-f(a)}{b_n-a}\right)\left(\frac{a-a_n}{b_n-a_n}\right)\right] = f'(a)$$

because $\lim\limits_{n\to\infty} \frac{f(b_n)-f(a)}{b_n-a} = \frac{f(a)-f(a_n)}{a-a_n} = f'(a)$ and $\left|\frac{a-a_n}{b_n-a_n}\right| \le 1 \ \forall n$.

b) For fixed $m \in \mathbb{N}\cup\{0\}$, x falls between adjacent dyadic points $\frac{p_m}{2^m} < x < \frac{p_m+1}{2^m}$,
so set $x_m = p_m/2^m$ and $y_m = (p_m+1)/2^m$.

$h_n(x_m) = h_n(y_m) = 0 \ \forall\, n > m$.

$\Rightarrow g_m(x_m) = g(x_m)$ and $g_m(y_m) = g(y_m)$

g_m is the line segment connecting $(x_m, g(x_m))$ and $(y_m, g(y_m))$

$\Rightarrow g_m'(x) = \dfrac{g(y_m)-g(x_m)}{y_m - x_m}$

Also, $|g_{m+1}'(x) - g_m'(x)| = |h_{m+1}'(x)|$ where $h_{m+1}'(x) = \pm 1$

$\Rightarrow g_m'(x)$ is not Cauchy and does not converge

$\Rightarrow \lim\limits_{m\to\infty} \dfrac{g(y_m)-g(x_m)}{y_m-x_m}$ DNE

$\Rightarrow g$ is not differentiable at x by (a).

Chapter 6

Sequences and Series of Functions

6.1 Discussion: The Power of Power Series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots < 2$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \ |x|<1 \Rightarrow \frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^4 + \cdots, \ |x|<1$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \text{ for } |x|<1 \Rightarrow \arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$(1+x)^r = 1 + rx + \frac{r(r-1)}{2!} x^2 + \frac{r(r-1)(r-2)}{3!} x^3 + \cdots$$

$$r = \frac{1}{2} \Rightarrow \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{2^2 2!} x^2 + \frac{3}{2^3 3!} x^3 - \frac{3 \cdot 5}{2^4 4!} x^4 + \cdots$$

$$(\sqrt{1+x})^2 = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots\right)\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots\right) = 1 + \left(\frac{1}{2} + \frac{1}{2}\right)x + \left(-\frac{1}{8} + \frac{1}{4} - \frac{1}{8}\right)x^2 + \cdots = 1 + x + 0x^2 + 0x^3 + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \Rightarrow \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

$$p(x) = 1 + ax + bx^2 + cx^3 = \left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right)\left(1 - \frac{x}{r_3}\right) \text{ for roots } r_1, r_2, r_3.$$

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\cdots = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots$$

$$= 1 + \left(-\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} - \cdots\right)x^2 + \left(\frac{1}{4\pi^4} + \frac{1}{9\pi^4} + \cdots\right)x^4 + \cdots$$

$$\Rightarrow -\frac{1}{3!} = -\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} - \cdots \Rightarrow \frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

6.2 Uniform Convergence of a Sequence of Functions

Definition 6.2.1. For each $n \in \mathbf{N}$, let f_n be a function defined on a set $A \subseteq \mathbf{R}$. The sequence (f_n) of functions converges pointwise on A to a function f if, for all $x \in A$, the sequence of real numbers $f_n(x)$ converges to $f(x)$. In this case, we write $f_n \to f$, $\lim f_n = f$, or $\lim_{n\to\infty} f_n(x) = f(x)$.

Example 1. (i) Consider

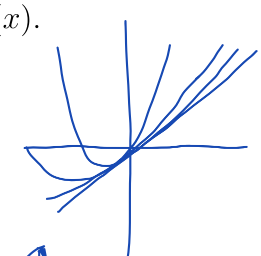
$$f_n(x) = (x^2 + nx)/n$$

on all of $\mathbf{R}$. Find $\lim_{n\to\infty} f_n(x)$.

(ii) Let $g_n(x) = x^n$ on the set $[0,1]$, and consider what happens as n tends to infinity.

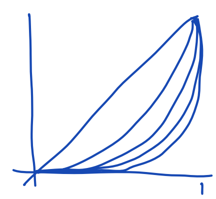
(iii) Consider $h_n(x) = x^{1+\frac{1}{2n-1}}$ on the set $[-1,1]$. Find $\lim_{n\to\infty} h_n(x)$.

(i) $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2+nx}{n} = \lim_{n\to\infty} \frac{x^2}{n} + x = x$



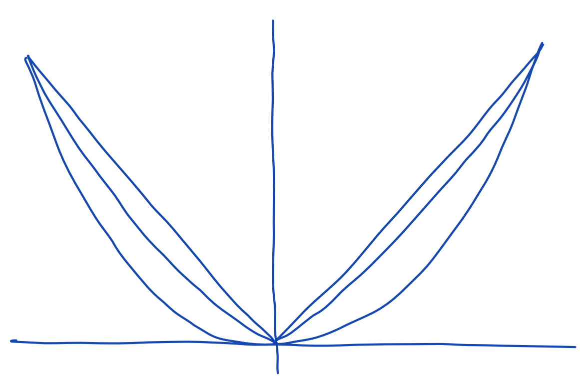
(ii) If $0 \le x < 1$ then $x^n \to 0$.
If $x=1$, then $x^n \to 1$.
$\Rightarrow g_n \to g$ pointwise on $[0,1]$ where

$$g(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1 \end{cases}$$



(iii) For fixed $x \in [-1,1]$, we have

$$\lim_{n\to\infty} h_n(x) = x \lim_{n\to\infty} x^{\frac{1}{2n-1}} = \lim_{n\to\infty} x^{1+\frac{1}{2n-1}} = \lim_{n\to\infty} x^{\frac{2n}{2n-1}} = \lim_{n\to\infty} (x^2)^{\frac{n}{2n-1}} = \sqrt{x^2} = |x|$$



Definition 6.2.2 (Uniform Convergence)**.** Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbf{R}$. Then, (f_n) converges uniformly on A to a limit function f defined on A if, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that $|f_n(x) - f(x)| < \epsilon$ whenever $n \geq N$ and $x \in A$.

Definition 6.2.1B. Let f_n be a sequence of functions defined on a set $A \subseteq \mathbf{R}$. Then, (f_n) converges pointwise on A to a limit f defined on A if, for every $\epsilon > 0$ and $x \in A$, there exists an $N \in \mathbf{N}$ (perhaps dependent on x such that $|f_n(x) - f(x)| < \epsilon$ whenever $n \geq N$.

Example 2. (i) Let

$$g_n(x) = \frac{1}{n(1 + x^2)}.$$

Does g_n converge uniformly on $\mathbf{R}$?

(ii) Does $f_n(x) = (x^2 + nx)/n$ converge uniformly on $\mathbf{R}$?

(i) For fixed $x \in \mathbb{R}$, $\lim g_n(x) = 0$.

$\frac{1}{1+x^2} \leq 1 \quad \forall x \in \mathbb{R} \Rightarrow |g_n(x) - g(x)| = \left| \frac{1}{n(1+x^2)} - 0 \right| \leq \frac{1}{n}$

Thus given $\varepsilon > 0$, we can choose $N > 1/\varepsilon$ so that

$n \geq N$ implies $|g_n(x) - g(x)| < \varepsilon \quad \forall x \in \mathbb{R}$

$\Rightarrow g_n \to 0$ uniformly on $\mathbb{R}$.

(ii) By Ex 1, $f_n \to f(x) = x$ pointwise on $\mathbb{R}$.

$|f_n(x) - f(x)| = \left| \frac{x^2 + nx}{n} - x \right| = \frac{x^2}{n}$

so for $|f_n(x) - f(x)| < \varepsilon$ we need to choose $N > \frac{x^2}{\varepsilon}$

no value of N will work for all values of x

$\Rightarrow f_n(x)$ does not converge uniformly on $\mathbb{R}$.

But on $[-b, b]$ we have $\frac{x^2}{n} \leq \frac{b^2}{n}$

So given $\varepsilon > 0$, we can choose $N > \frac{b^2}{\varepsilon}$

$\Rightarrow f_n \to f$ uniformly on $[-b, b]$

Theorem 6.2.1 (Cauchy Criterion for Uniform Convergence)**.** *A sequence of functions (f_n) defined on a set $A \subseteq \mathbf{R}$ converges uniformly on A if and only if for every $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ whenever $m, n \geq N$ and $x \in A$.*

Example 3. Prove Theorem 6.2.1.

($\Rightarrow$) Let $\varepsilon > 0$. $f_n \to f$ uniformly $\Rightarrow f(x) = \lim\limits_{n\to\infty} f_n(x)$

$\Rightarrow \exists N$ s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \forall\, n \geq N$ and $x \in A$

Then given $m, n \geq N$,
$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\, x \in A \end{aligned}$$

($\Leftarrow$) For each $x \in A$ $(f_n(x))$ is Cauchy.

$\Rightarrow (f_n(x))$ converges

$\Rightarrow f(x) = \lim\limits_{n\to\infty} f_n(x)$, i.e., $f_n(x) \to f(x)$ pointwise on A.

Let $\varepsilon > 0$. Then $\exists\, N$ s.t. $-\varepsilon < f_n(x) - f_m(x) < \varepsilon \quad \forall\, m, n \geq N$ and $x \in A$.

$\Rightarrow \lim\limits_{m\to\infty} (f_n(x) - f_m(x)) = f_n(x) - f(x)$ for each $x \in A$.

$\Rightarrow -\varepsilon \leq f_n(x) - f(x) \leq \varepsilon \quad \forall\, n \geq N$ and $x \in A$.

$\Rightarrow f_n \to f$ uniformly on A.

Theorem 6.2.2 (Continuous Limit Theorem). *Let (f_n) be a sequence of functions defined on $A \subseteq \mathbf{R}$ that converges uniformly on A to a function f. If each f_n is continuous at $c \in A$, then f is continuous at c.*

Proof. Fix $c \in A$ and let $\epsilon > 0$. Choose N so that

$$|f_N(x) - f(x)| < \frac{\epsilon}{3}$$

for all $x \in A$. Because f_N is continuous, there exists a $\delta > 0$ for which

$$|f_N(x) - f_N(c)| < \frac{\epsilon}{3}$$

is true whenever $|x - c| < \delta$. But this implies

$$\begin{aligned}
|f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\
&\le |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Thus, f is continuous at $c \in A$. □

Example 4. Recall that the Bolzano–Weierstrass Theorem (Theorem 2.5.2) states that every bounded sequence of real numbers has a convergent subsequence. An analogous statement for bounded sequences of functions is not true in general, but under stronger hypotheses several different conclusions are possible. One avenue is to assume the common domain for all of the functions in the sequence is countable. (Another is explored in the next two examples.)

Let $A = \{x_1, x_2, x_3, \ldots\}$ be a countable set. For each $n \in \mathbf{N}$, let f_n be defined on A and assume there exists an $M > 0$ such that $|f_n(x)| \le M$ for all $n \in \mathbf{N}$ and $x \in A$. Follow these steps to show that there exists a subsequence of (f_n) that converges pointwise on A.

(a) Why does the sequence of real numbers $f_n(x_1)$ necessarily contain a convergent subsequence (f_{n_k})? To indicate that the subsequence of functions (f_{n_k}) is generated by considering the values of the functions at x_1, we will use the notation $f_{n_k} = f_{1,k}$.

(b) Now, explain why the sequence $f_{1,k}(x_2)$ contains a convergent subsequence.

(c) Carefully construct a nested family of subsequences $(f_{m,k})$, and show how this can be used to produce a single subsequence of (f_n) that converges at every point of A.

a) $f_n(x_1)$ is bounded by $M \Rightarrow \exists$ a convergent subsequence by Bolzano-Weierstrass

b) $f_{1,k}(x_2)$ has a convergent subsequence $f_{2,k}(x_2)$ by Bolzano-Weierstrass

c) If $m' > m$ then $(f_{m',k})$ is a subsequence of $(f_{m,k})$.

Let $f_{n_k} = f_{k,k} = (f_{1,1}, f_{2,2}, f_{3,3}, \ldots)$

Then $(f_{k,k})$ is a subsequence of $f_{1,k}$ and thus $f_{k,k}(x_1)$ converges.

After m terms $f_{k,k}$ is a subsequence of $f_{m,k}$.

$\Rightarrow f_{k,k}(x_m)$ converges for $x_m \in A$

$\Rightarrow f_{k,k}$ converges pointwise on A.

Example 5. A sequence of functions (f_n) defined on a set $E \subseteq \mathbf{R}$ is called equicontinuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all $n \in \mathbf{N}$ and $|x - y| < \delta$ in E.

(a) What is the difference between saying that a sequence of functions (f_n) is equicontinuous and just asserting that each f_n in the sequence is individually uniformly continuous?

(b) Give a qualitative explanation for why the sequence $g_n(x) = x^n$ is not equicontinuous on $[0, 1]$. Is each g_n uniformly continuous on $[0, 1]$?

a) If each f_n is uniformly continuous δ will not depend on x but does depend on f_n.

b) For each n, g_n is continuous on $[0,1] \Rightarrow g_n$ is uniformly continuous.

Take $\varepsilon = 1/2$ and $y = 1$. Equicontinuity requires $\delta > 0$ s.t.

$$|x^n - 1| < \tfrac{1}{2} \quad \forall\, n \in \mathbb{N} \text{ and } |x - 1| < \delta$$

But δ cannot be independent of n, since there will always be n s.t. $|x^n - 1| \geq 1/2$, no matter how close x is to 1.

Example 6 (Arzela–Ascoli Theorem)**.** For each $n \in \mathbf{N}$, let f_n be a function defined on $[0, 1]$. If (f_n) is bounded on $[0, 1]$—that is, there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all $n \in \mathbf{N}$ and $x \in [0, 1]$—and if the collection of functions (f_n) is equicontinuous, follow these steps to show that (f_n) contains a uniformly convergent subsequence.

(a) Use Example 4 to produce a subsequence (f_{n_k}) that converges at every rational point in $[0, 1]$. To simplify the notation, set $g_k = f_{n_k}$. It remains to show that (g_k) converges uniformly on all of $[0, 1]$.

(b) Let $\epsilon > 0$. By equicontinuity, there exists a $\delta > 0$ such that

$$|g_k(x) - g_k(y)| < \frac{\epsilon}{3}$$

for all $|x - y| < \delta$ and $k \in \mathbf{N}$. Using this δ, let $r_1, r_2, \dots, r_m$ be a *finite* collection of rational points with the property that the union of the neighborhoods $V_\delta(r_i)$ contains $[0, 1]$.
Explain why there must exist an $N \in \mathbf{N}$ such that

$$|g_s(r_i) - g_t(r_i)| < \frac{\epsilon}{3}$$

for all $s, t \geq N$ and r_i in the finite subset of $[0, 1]$ just described. Why does having the set $\{r_1, r_2, \dots, r_m\}$ be finite matter?

(c) Finish the argument by showing that, for an arbitrary $x \in [0, 1]$,

$$|g_s(x) - g_t(x)| < \epsilon$$

for all $s, t \geq N$.

a) $[0,1]$ is countable, so (g_K) exists by Ex. 4.

b) Let r_i be fixed. (g_K) converges pointwise at every rational $\Rightarrow (g_K(r_i))$ is Cauchy
$\Rightarrow \exists N_i$ s.t. $|g_s(r_i) - g_t(r_i)| < \frac{\varepsilon}{3} \ \forall s,t \geq N_i$.
Let $N = \max\{N_1, N_2, \dots, N_m\}$.
If $\{r_1, r_2, \dots, r_m\}$ were infinite then N would be the max of an infinite set.

c) For $x \in [0,1] \ \exists r_i$ s.t. $|r_i - x| < \delta$
$\Rightarrow |g_s(x) - g_s(r_i)| < \frac{\varepsilon}{3} \ \forall s \in \mathbb{N}$
$$\Rightarrow |g_s(x) - g_t(x)| = |g_s(x) - g_s(r_i) + g_s(r_i) - g_t(r_i) + g_t(r_i) - g_t(x)|$$
$$\leq |g_s(x) - g_s(r_i)| + |g_s(r_i) - g_t(r_i)| + |g_t(r_i) - g_t(x)|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \forall s,t \geq N$$

6.3 Uniform Convergence and Differentiation

Theorem 6.3.1 (Differentiable Limit Theorem)**.** *Let $f_n \to f$ pointwise on the closed interval $[a, b]$, and assume that each f_n is differentiable. If (f_n') converges uniformly on $[a, b]$ to a function g, then the function f is differentiable and $f' = g$.*

Proof. Fix $c \in [a, b]$ and let $\epsilon > 0$. We want to argue that $f'(c)$ exists and equals $g(c)$. Because f' is defined by the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

our task is to produce a $\delta > 0$ so that

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \epsilon$$

whenever $0 < |x - c| < \delta$.

To motivate the strategy of the proof, observe that for all $x \neq c$ and all $n \in \mathbf{N}$, the triangle inequality implies

$$\begin{aligned} \left| \frac{f(x) - f(c)}{x - c} \right| \leq & \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \\ & + \left| \frac{f_n(x) - f_n(c)}{x - c} - f_n'(c) \right| + |f_n'(c) - g(c)|. \end{aligned}$$

Our intent is to first find an f_n that forces the first and third terms on the right-hand side to be less than $\epsilon/3$. Once we establish which f_n we want, we can then use the differentiability of f_n to produce a δ that makes the middle term less than $\epsilon/3$ for all x satisfying $0 < |x - c| < \delta$.

Let's start by choosing an N_1 such that

$$|f_m'(c) - g(c)| < \frac{\epsilon}{3} \tag{1}$$

for all $m \geq N_1$. We now invoke the uniform convergence of (f_n') to assert (via Theorem 6.2.1) that there exists an N_2 such that $m, n \geq N_2$ implies

$$|f_m'(x) - f_n'(x)| < \frac{\epsilon}{3} \quad \text{for all } x \in [a, b].$$

Set $N = \max\{N_1, N_2\}$.

The function f_N is differentiable at c, and so there exists a $\delta > 0$ for which

$$\left| \frac{f_N(x) - f_N(c)}{x - c} - f_N'(c) \right| < \frac{\epsilon}{3} \tag{2}$$

whenever $0 < |x - c| < \delta$. This is our sought after δ, but it takes some effort to show that it has the desired property.

Fix an x satisfying $0 < |x - c| < \delta$, let $m \geq N$, and apply the Mean Value Theorem to $f_m - f_N$ on the interval $[c, x]$, (If $x < c$ the argument is the same.) By MVT, there exists an $\alpha \in (c, x)$ such that

$$f'_m(\alpha) - f'_N(\alpha) = \frac{(f_m(x) - f_N(x)) - (f_m(c) - f_N(c))}{x - c}.$$

Recall that our choice of N implies

$$|f'_m(\alpha) - f'_N(\alpha)| < \frac{\epsilon}{3},$$

and so it follows that

$$\left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| < \frac{\epsilon}{3}.$$

Because $f_m \to f$ we can take the limit as $m \to \infty$, and the Order Limit Theorem (Theorem 2.3.3) asserts that

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| \leq \frac{\epsilon}{3}. \tag{3}$$

Finally, the inequalities in (1), (1), and (1) together imply that for x satisfying $0 < |x - c| < \delta$,

$$\begin{aligned} \left| \frac{f(x) - f(c)}{x - c} - g(c) \right| &\leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| \\ &\quad + \left| \frac{f_N(x) - f_N(c)}{x - c} - f'_N(c) \right| + |f'_N(c) - g(c)|. \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

□

Theorem 6.3.2. *Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a, b]$, and assume (f'_n) converges uniformly on $[a, b]$. If there exists a point $x_0 \in [a, b]$ where $f_n(x_0)$ is convergent, then (f_n) converges uniformly on $[a, b]$.*

Example 1. Prove Theorem 6.3.2.

Let $x \in [a,b]$ and assume WLOG that $x > x_0$.

By MVT $\exists\, \alpha$ s.t. $(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0)) = (f_n'(\alpha) - f_m'(\alpha))(b-a)$

Let $\varepsilon > 0$. (f_n') converges uniformly $\Rightarrow \exists\, N_1$ s.t.

$$|f_n'(c) - f_m'(c)| < \frac{\varepsilon}{2(b-a)} \quad \forall\, n,m \geq N \text{ and } c \in [a,b]$$

$(f_n(x_0))$ converges $\Rightarrow \exists\, N_2$ s.t.

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2} \quad \forall\, n,m \geq N_2$$

Let $N = \max\{N_1, N_2\}$. Then if $n,m \geq N$ it follows that

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| \\ &= |(f_n'(\alpha) - f_m'(\alpha))(b-a)| + |f_n(x_0) - f_m(x_0)| \\ &< \frac{\varepsilon}{2(b-a)}(b-a) + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

N doesn't depend on x, so (f_n) converges uniformly on $[a,b]$.

Theorem 6.3.3. *Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a,b]$, and assume (f_n') converges uniformly to a function g on $[a,b]$. If there exists a point $x_0 \in [a,b]$ for which $f_n(x_0)$ is convergent, then (f_n) converges uniformly. Moreover, the limit function $f = \lim f_n$ is differentiable and satisfies $f' = g$.*

Example 2. Let
$$g_n(x) = \frac{nx + x^2}{2n},$$
and set $g(x) = \lim g_n(x)$. Show that g is differentiable in two ways:

(a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.

(b) Compute $g_n'(x)$ for each $n \in \mathbf{N}$ and show that the sequence of derivatives (g_n') converges uniformly on every interval $[-M, M]$. Use Theorem 6.3.3 to conclude $g'(x) = \lim g_n'(x)$.

(c) Repeat parts (a) and (b) for the sequence $f_n(x) = (nx^2 + 1)/(2n + x)$.

a) $g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{x}{2} + \frac{x^2}{2n} = \frac{x}{2} \Rightarrow g'(x) = \frac{1}{2}$

b) $g_n'(x) = \frac{1}{2} + \frac{x}{n} \longrightarrow \frac{1}{2}$

For $x \in [-M, M]$, $|g_n'(x) - \frac{1}{2}| = \left|\frac{x}{n}\right| \le \frac{M}{n}$

Given $\varepsilon > 0$, choose $N > M/\varepsilon$. Then $n \ge N$ implies $|g_n'(x) - \frac{1}{2}| < \varepsilon$

$\Rightarrow g_n' \to 1/2$ uniformly on $[-M, M] \Rightarrow g'(x) = \frac{1}{2}$

c) $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2 + 1/n}{2 + x/n} = \frac{x^2}{2} \Rightarrow f'(x) = x$

$$f_n'(x) = \frac{4n^2x + 3nx^2 + 1}{4n^2 + 4nx + x^2}$$

$$\Rightarrow \lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{4x + 3x^2/n + 1/n^2}{4 + 4x/n + x^2/n^2} = x$$

For $|x| < M$, we have

$$|f_n'(x) - x| = \left|\frac{-nx^2 - x^3 + 1}{4n^2 + 4nx + x^2}\right| \le \frac{nM^2 + M^3 + 1}{4n^2 - 4nM} \longrightarrow 0 \quad \text{when } n > M.$$

$\Rightarrow f_n'(x) \to x$ uniformly on $[-M, M]$

6.4 Series of Functions

Definition 6.4.1. For each $n \in \mathbf{N}$, let f_n be functions defined on a set $A \subseteq \mathbf{R}$. The infinite series

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \cdots$$

converges pointwise on A to $f(x)$ if the sequence $s_k(x)$ of partial sums defined by

$$s_k(x) = f_1(x) + f_2(x) + \cdots + f_k(x)$$

converges pointwise to $f(x)$. The series converges uniformly on A to f is the sequence $s_k(x)$ converges uniformly on A to $f(x)$.

In either case, we write $f = \sum_{n=1}^{\infty} f_n$ or $f(x) = \sum_{n=1}^{\infty} f_n(x)$, always being explicit about the type of convergence involved.

Theorem 6.4.1 (Term-by-term Continuity Theorem). *Let f_n be continuous functions defined on set $A \subseteq \mathbf{R}$, and assume $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to a function f. Then, f is continuous on A.*

Proof. Apply the Continuous Limit Theorem (Theorem 6.2.2 to the partial sums $s_k = f_1 + f_2 + \cdots + f_k$. $\square$

Theorem 6.4.2 (Term-by-term Differentiability Theorem). *Let f_n be differentiable functions defined on an interval A, and assume $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly to a limit $g(x)$ on A. If there exists a point $x_0 \in [a, b]$ where $\sum_{n=1}^{\infty} f_n(x_0)$ converges, then the series $\sum_{n=1}^{\infty}$ converges uniformly to a differentiable function $f(x)$ satisfying $f'(x) = g(x)$ on A. In other words,*

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{and} \quad f'(x) = \sum_{n=1}^{\infty} f_n'(x).$$

Proof. Apply the stronger form of the Differentiable Limit Theorem (Theorem 6.3.3) to the partial sums $s_k = f_1 + f_2 + \cdots + f_k$. Observe that Theorem 5.2.2 implies that $s_k' = f_1' + f_2' + \cdots + f_k'$. $\square$

Theorem 6.4.3 (Cauchy Criterion for Uniform Convergence of Series). *A series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $A \subseteq \mathbf{R}$ if and only if for every $\epsilon > 0$ there exists an $n \in \mathbf{N}$ such that*

$$|f_{m+1}(x) + f_{m+2}(x) + f_{m+3}(x) + \cdots + f_n(x)| < \epsilon$$

whenever $n > m \geq N$ and $x \in A$.

Corollary 6.4.1 (Weierstrass M-Test)**.** *For each $n \in \mathbf{N}$, let f_n be a function defined on a set $A \subseteq \mathbf{R}$, and let $M_n > 0$ be a real number satisfying*

$$|f_n(x)| \leq M_n$$

for all $x \in A$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

Example 1. Prove Corollary 6.4.1.

Let $\varepsilon > 0$.

$\sum_{n=1}^{\infty} M_n$ converges $\Rightarrow \exists\ N$ s.t. $n > m \geq N$ implies

$$M_{m+1} + M_{m+2} + \cdots + M_n < \varepsilon$$

Then $|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| \leq M_{m+1} + M_{m+2} + \cdots + M_n$

$\Rightarrow \sum_{n=1}^{\infty} f_n$ converges uniformly by the Cauchy criterion

Example 2. (a) Show that

$$g(x) = \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n}$$

is continuous on all of $\mathbf{R}$.

(b) The function g is an example of a continuous nowhere differentiable function. What happens if we try to use Theorem 6.4.2 to explore whether g is differentiable?

a) $\left|\frac{\cos(2^n x)}{2^n}\right| \leq \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{\cos(2^n x)}{2^n}$ converges uniformly by the Weierstrass M-Test

$\Rightarrow g$ is continuous by Theorem 6.4.1.

b) Term by term differentiation gives $\sum_{n=0}^{\infty} -\sin(2^n x)$.

We can't use the Weierstrass M-Test without $\frac{1}{2^n}$ (the terms don't go to zero in general)

$\Rightarrow \sum_{n=0}^{\infty} -\sin(2^n x)$ does not converge uniformly

$\Rightarrow$ We cannot apply Theorem 6.4.2.

Example 3. (a) Prove that

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \cdots$$

is continuous on $[-1, 1]$.

(b) The series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

converges for every x in the half-open interval $[-1, 1)$ but does not converge when $x = 1$. For a fixed $x_0 \in (-1, 1)$, explain how we can still use the Weierstrass M-Test to prove that f is continuous at x_0.

a) For $x \in [-1,1]$, we have $\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges uniformly by the Weierstrass M-Test

$\Rightarrow h$ is continuous by Theorem 6.4.1.

b) Fix $x_0 \in (-1,1)$ and choose c s.t. $|x_0| < c < 1$.

On $[-c,c]$, we have $\left|\frac{x^n}{n}\right| \leq \frac{c^n}{n}$ and $\sum_{n=1}^{\infty} \frac{c^n}{n}$ converges.

$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{n}$ converges uniformly on $[-c,c]$ by the Weierstrass M-Test

$\Rightarrow h$ is continuous at $x_0 \in [-c,c]$ by Theorem 6.4.1.

6.5 Power Series

Theorem 6.5.1. *If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \in \mathbf{R}$, then it converges absolutely for any x satisfying $|x| < |x_0|$.*

Proof. If $\sum_{n=0}^{\infty} a_n x_0^n$ converges, then the sequence of terms $(a_n x_0^n)$ is bounded. (In fact, it converges to 0.) Let $M > 0$ satisfy $|a_n x_0^n| \le M$ for all $n \in \mathbf{N}$. If $x \in \mathbf{R}$ satisfies $|x| < |x_0|$, then

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \le M \left|\frac{x}{x_0}\right|^n.$$

But notice that

$$\sum_{n=0}^{\infty} M \left|\frac{x}{x_0}\right|^n$$

is a geometric series with ratio $|x/x_0| < 1$ and so converges. By the Comparison Test, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely. □

Theorem 6.5.2. *If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at a point x_0, then it converges uniformly on the closed interval $[-c, c]$, where $c = |x_0|$.*

Example 1. Prove Theorem 6.5.2.

Set $M_n = |a_n x_0^n|$. Then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at x_0

$$\Rightarrow \sum_{n=0}^{\infty} |a_n x_0^n| = \sum_{n=0}^{\infty} M_n \text{ converges.}$$

If $x \in [-c, c]$ then $|a_n x^n| \le |a_n x_0^n| = M_n$

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-c, c]$ by the Weierstrass M-Test.

***Lemma* 6.5.1** (Abel's Lemma)**:** Let b_n satisfy $b_1 \ge b_2 \ge b_3 \ge \cdots \ge 0$, and let $\sum_{n=1}^{\infty} a_n$ be a series for which the partial sums are bounded. In other words, assume there exists $A > 0$ such that

$$|a_1 + a_2 + \cdots + a_n| \le A$$

for all $n \in \mathbf{N}$. Then, for all $n \in \mathbf{N}$,

$$|a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n| \le A b_1.$$

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$. Using the summation-by-parts formula derived in Example 3 of Section 7.2, we can write

$$\begin{aligned}\left|\sum_{k=1}^{n} a_k b_k\right| &= \left|s_n b_{n+1} + \sum_{k=1}^{n} s_k(b_k - b_{k+1})\right| \\ &\le A b_{n+1} + \sum_{k=1}^{n} A(b_k - b_{k+1}) \\ &= A b_{n+1} + (Ab_1 - Ab_{n+1}) = Ab_1. \end{aligned}$$

□

Theorem 6.5.3 (Abel's Theorem). *Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series that converges at the point $x = R > 0$. Then the series converges uniformly on the interval $[0, R]$. A similar result holds if the series converges at $x = -R$.*

Proof. To set the stage for an application of Lemma 6.5.1, we first write

$$g(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (a_n R^n) \left(\frac{x}{R}\right)^n.$$

Let $\epsilon > 0$. By the Cauchy Criterion for Uniform Convergence of Series (Theorem 6.2.1), we will be done if we can produce an N such that $n > m \ge N$ implies

$$\begin{aligned}&\left|(a_{m+1}R^{m+1})\left(\frac{x}{R}\right)^{m+1} + (a_{m+2}R^{m+2})\left(\frac{x}{R}\right)^{m+2} + \cdots\right. \\ &\qquad\qquad \left. + (a_n R^n)\left(\frac{x}{R}\right)^n\right| < \epsilon. \end{aligned} \tag{1}$$

Because we are assuming that $\sum_{n=0}^{\infty} a_n R^n$ converges, the Cauchy criterion for convergent series of real numbers guarantees that there exists an N such that

$$|a_{m+1}R^{m+1} + a_{m+2}R^{m+2} + \cdots + a_n R^n| < \frac{\epsilon}{2}$$

whenever $n > m \ge N$. But now, for any fixed $m \in \mathbf{N}$, we can apply Abel's Lemma 6.5.1 to the sequences obtained by omitting the first m terms. Using $\epsilon/2$ as a bound on the partial sums of $\sum_{j=1}^{\infty} a_{m+j} R^{m+j}$ and observing that $(x/R)^{m+j}$ is monotone decreasing, an application of Abel's Lemma to equation (1) yields

$$\begin{aligned}&\left|(a_{m+1}R^{m+1})\left(\frac{x}{R}\right)^{m+1} + (a_{m+2}R^{m+2})\left(\frac{x}{R}\right)^{m+2} + \cdots\right. \\ &\qquad\qquad \left. + (a_n R^n)\left(\frac{x}{R}\right)^n\right| \le \frac{\epsilon}{2}\left(\frac{x}{R}\right)^{m+1} < \epsilon. \end{aligned}$$

□

Theorem 6.5.4. *If a power series converges pointwise on the set $A \subseteq \mathbf{R}$, then it converges uniformly on any compact set $K \subseteq A$.*

Proof. A compact set contains both a maximum x_1 and a minimum x_0, which by hypothesis must be in A. Abel's Theorem implies the series converges uniformly on the interval $[x_0, x_1]$ and thus also on K. $\square$

Theorem 6.5.5. *If $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$, then the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges at each $x \in (-R, R)$ as well. Consequently, the convergence is uniform on compact sets contained in $(-R, R)$.*

Example 2. (a) If s satisfies $0 < s < 1$, show ns^{n-1} is bounded for all $n \geq 1$.

(b) Given an arbitrary $x \in (-R, R)$, pick t to satisfy $|x| < t < R$. Use this start to construct a proof for Theorem 6.5.5.

a) Set $y_n = ns^{n-1}$. If $n > \frac{s}{1-s}$, then $\frac{y_{n+1}}{y_n} = \frac{s(n+1)}{n} < 1 \Rightarrow (y_n)$ is eventually decreasing

$\Rightarrow (y_n)$ is bounded

b) Let $x \in (-R, R)$ be arbitrary and pick t s.t. $|x| < t < R$.

$$\sum_{n=1}^{\infty} |na_n x^{n-1}| = \sum_{n=1}^{\infty} \frac{1}{t}\left(n\left|\frac{x}{t}\right|^{n-1}\right)|a_n t^n|$$

$\left|\frac{x}{t}\right| < 1 \Rightarrow \exists L$ s.t. $n\left|\frac{x}{t}\right|^{n-1} \leq L \quad \forall n \in \mathbb{N}$

$$\Rightarrow \sum_{n=1}^{\infty} |na_n x^{n-1}| = \sum_{n=1}^{\infty} \frac{1}{t}\left(n\left|\frac{x}{t}\right|^{n-1}\right)|a_n t^n| \leq \frac{L}{t} \sum_{n=1}^{\infty} |a_n t^n|,$$

which converges because $t \in (-R, R)$

$\Rightarrow \sum_{n=1}^{\infty} na_n x^{n-1}$ converges absolutely

Theorem 6.5.6. *Assume*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges on an interval $A \subseteq \mathbf{R}$. The function f is continuous on A and differentiable on any open interval $(-R, R) \subseteq A$. The derivative is given by

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Moreover, f is infinitely differentiable on $(-R, R)$, and the successive derivatiives can be obtained via term-by-term differentiation of the appropriate series.

Proof. The details for why f is continuous have been discussed. Theorem 6.5.5 justifies the application of the Term-by-term Differentiability Theorem (Theorem 6.4.2), which verifies the formula for f'.

A differentiated power series is a power series in its own right, and Theorem 6.5.5 implies that, although the series may no longer converge at a particular endpoint, the radius of convergence does not change. By induction, then, power series are differentiable an infinite number of times. □

Example 3. If both $\sum a_n$ and $\sum b_n$ converge conditionally to A and B respectively, then it is possible for the Cauchy product,

$$\sum d_n \quad \text{where} \quad d_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0,$$

to diverge. However, if $\sum d_n$ does converge, then it must converge to AB. To prove this, set

$$f(x) = \sum a_n x^n, \quad g(x) = \sum b_n x^n, \quad \text{and} \quad h(x) = \sum d_n x^n.$$

Use Abel's Theorem and the result in Example 4 of section 2.8 to establish this result.

$\sum a_n$, $\sum b_n$, and $\sum d_n$ converge $\Rightarrow$ $f, g,$ and h converge on $[0,1]$

$\Rightarrow$ $f, g,$ and h are continuous on $[0,1]$

Fix $x \in [0,1)$.

Convergence at $1 \Rightarrow f, g,$ and h converge absolutely by Theorem 6.5.1

$\Rightarrow h(x) = \sum d_n x^n = f(x)g(x)$ by Ex. 2.8.4

This is true $\forall x \in [0,1)$ and $f, g,$ and h are continuous on $[0,1]$

$\Rightarrow h(1) = f(1)g(1)$

$\Rightarrow \sum d_n = \left(\sum a_n\right)\left(\sum b_n\right)$

Example 4. A series $\sum_{n=0}^{\infty} a_n$ is said to be Abel-summable to L if the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for all $x \in [0, 1)$ and $L = \lim_{x \to 1^-} f(x)$.

(a) Show that any series that converges to a limit L is also Abel-summable to L.

(b) Show that $\sum_{n=0}^{\infty} (-1)^n$ is Abel-summable and find the sum.

a) Assume $\sum a_n$ converges to L.

Then $f(x) = \sum a_n x^n$ converges uniformly on $[0,1]$.

$\Rightarrow f$ is continuous on $[0,1]$

$\Rightarrow \lim_{x \to 1^-} f(x) = f(1) = L$

b) $$\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 -$$

$$= \frac{1}{1-(-x)}$$

$$= \frac{1}{1+x} \quad \text{for } |x| < 1$$

$\lim_{x \to 1^-} \frac{1}{1+x} = \frac{1}{2} \Rightarrow \sum_{n=0}^{\infty} (-1)^n$ is Abel-summable to $\frac{1}{2}$

6.6 Taylor Series

Example 1. Find series representations for $1/(1-x)^2$ and $\arctan(x)$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad \text{for } |x|<1$$

$$\Rightarrow \frac{1}{(1-x^2)} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots \quad \text{for } |x|<1$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \quad \text{for } |x|<1$$

$$\Rightarrow \arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots \quad \text{for } |x|<1$$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots \quad |x-a|<R$$

$$\Rightarrow f(a) = c_0$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \quad |x-a|<R$$

$$\Rightarrow f'(a) = c_1$$

$$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + \cdots \quad |x-a|<R$$

$$\Rightarrow f''(a) = 2c_2$$

$$f'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + 3\cdot 4\cdot 5c_5(x-a)^2 + \cdots \quad |x-a|<R$$

$$\Rightarrow f'''(a) = 2\cdot 3c_3 = 3!\,c_3$$

$$f^{(n)}(a) = 2\cdot 3\cdot 4\cdot \cdots \cdot n\,c_n = n!\,c_n$$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Theorem 6.6.1 (Taylor's Formula)**.** *Let*

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

be defined on some nontrivial interval centered at zero. Then,

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

Example 2. Prove Theorem 6.6.1.

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots \quad |x|<R \quad \Rightarrow f(0) = a_0$$

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots \quad |x|<R \quad \Rightarrow f'(0) = a_1$$

$$f''(x) = 2a_2 + 2\cdot 3a_3x + 3\cdot 4a_4x^2 + 4\cdot 5a_5x^3 + \cdots \quad |x|<R \quad \Rightarrow f''(0) = 2a_2$$

$$f'''(x) = 2\cdot 3a_3 + 2\cdot 3\cdot 4a_4x + 3\cdot 4\cdot 5a_5x^2 + \cdots \quad |x|<R \quad \Rightarrow f'''(0) = 2\cdot 3a_3 = 3!\,a_3$$

$$\vdots$$

$$f^{(n)}(0) = 2\cdot 3\cdot 4\cdot \cdots \cdot n\,a_n = n!\,a_n$$

$$\Rightarrow a_n = \frac{f^{(n)}(0)}{n!}$$

Theorem 6.6.2 (Lagrange's Remainder Theorem)**.** *Let f be differentiable $N+1$ times on $(-R, R)$, define $a_n = f^{(n)}(0)/n!$ for $n = 0, 1, \ldots, N$, and let*

$$S_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N.$$

Given $x \neq 0$ in $(-R, R)$, there exists a point c satisfying $|c| < |x|$ where the error function $E_N(x) = f(x) - S_N(x)$ satisfies

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}.$$

Example 3. Show that the Taylor series for $\sin(x)$ converges uniformly to $\sin(x)$ on every interval of the form $[-R, R]$ for an arbitrary constant R.

$a_0 = \sin(0) = 0$, $a_1 = \cos(0) = 1$, $a_2 = -\sin(0)/2! = 0$, $a_3 = -\cos(0)/3! = -1/3!$

$S_N(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$|f^{(N+1)}(c)| \leq 1$

$\Rightarrow E_N(x) = \left| \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1} \right| \leq \frac{1}{(N+1)!} R^{N+1}$ for $x \in [-R, R]$

$\Rightarrow E_N(x) \to 0$ uniformly on $[-R, R]$

Proof of Lagrange's Remainder Theorem. The Taylor coefficients are chosen so that the function f and the polynomial S_N have the same derivatives at zero, at least up through the Nth derivative, after which S_N becomes the zero function. In other words, $f^{(n)}(0) = S_N^{(n)}(0)$ for all $0 \leq n \leq N$, which implies the error function $E_N(x) = f(x) - S_N(x)$ satisfies

$$E_N^{(n)}(0) = 0 \quad \text{for all } n = 0, 1, 2, \ldots, N.$$

To simplify notation, let's assume $x > 0$ and apply the Generalized Mean Value Theorem (Theorem 5.3.3) to the functions $E_N(x)$ and x^{N+1} on the interval $[0, x]$. Thus, there exists a point $x_1 \in (0, x)$ such that

$$\frac{E_N(x)}{x^{N+1}} = \frac{E_N'(x_1)}{(N+1)x_1^N}.$$

Now apply the Generalized Mean Value Theorem to the functions $E_N'(x)$ and $(N+1)x^N$ on the interval $[0, x_1]$ to get that there exists a point $x_2 \in (0, x_1)$ where

$$\frac{E_N(x)}{x^{N+1}} = \frac{E_N'(x_1)}{(N+1)x_1^N} = \frac{E_N''(x_2)}{(N+1)Nx_2^{N-1}}.$$

Continuing in this manner we find

$$\frac{E_N(x)}{x^{N+1}} = \frac{E_N^{(N+1)}(x_{N+1})}{(N+1)!}$$

where $x_{N+1} \in (0, x_N) \subseteq \cdots \subseteq (0, x)$. Now set $c = x_{N+1}$. Because $S_N^{(N+1)}(x) = 0$, we have $E_N^{(N+1)}(x) = f^{(N+1)}(x)$ and it follows that

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$$

as desired. $\square$

Example 4 (Cauchy's Remainder Theorem)**.** Let f be differentiable $N+1$ times on $(-R, R)$. For each $a \in (-R, R)$, let $S_N(x, a)$ be the partial sum of the Taylor series for f centered at a; in other words, define

$$S_N(x, a) = \sum_{n=0}^{N} c_n (x-a)^n \quad \text{and} \quad c_n = \frac{f^{(n)}(a)}{n!}.$$

Let $E_N(x, a) = f(x) - S_N(x, a)$. Now fix $x \neq 0$ in $(-R, R)$ and consider $E_N(x, a)$ as a function of a.

(a) Find $E_N(x, x)$.

(b) Explain why $E_N(x, a)$ is differentiable with respect to a, and show

$$E'_N(x, a) = \frac{-f^{(N+1)}(a)}{N!}(x-a)^N.$$

(c) Show

$$E_N(x) = E_N(x, 0) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x$$

for some c between 0 and x. This is Cauchy's form of the remainder for Taylor series centered at the origin.

a) $E_N(x,x) = f(x) - S_N(x,x) = f(x) - f(x) = 0$

b) $S_N(x,a) = f(a) + \sum_{n=1}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is differentiable

$\Rightarrow E_N(x,a)$ is differentiable

$$\begin{aligned}
E_N'(x,a) &= (f(x) - S_N(x,a))' \\
&= -S_N'(x,a) \\
&= -f'(a) - \sum_{n=1}^{N} \frac{f^{(n)}(a)}{n!} n(x-a)^{n-1}(-1) + \frac{f^{(n+1)}(a)}{n!}(x-a)^n \\
&= -f'(a) + \sum_{n=1}^{N} \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1} - \frac{f^{(n+1)}(a)}{n!}(x-a)^n \\
&= -f'(a) + \left(f'(a) - \frac{f^{(N+1)}(a)}{N!}(x-a)^N\right) \\
&= -\frac{f^{(N+1)}(a)}{N!}(x-a)^N
\end{aligned}$$

c) Applying MVT to $E_N(x,a)$ on $[0,x]$ gives

$$\frac{E_N(x,x) - E_N(x,0)}{x-0} = E_N'(x,c) \quad \text{for some } c \in (0,x)$$

$$\Rightarrow E_N(x) = E_N(x,0) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x \quad \text{by (b)}$$

6.7 The Weierstrass Approximation Theorem

Theorem 6.7.1 (Weierstrass Approximation Theorem)**.** *Let $f : [a,b] \to \mathbf{R}$ be continuous. Given $\epsilon > 0$, there exists a polynomial $p(x)$ satisfying*

$$|f(x) - p(x)| < \epsilon$$

for all $x \in [a,b]$.

Example 1. Assuming the Weierstrass Approximation Theorem (WAT), show that if f is continuous on $[a,b]$, then there exists a sequence (p_n) of polynomials such that $p_n \to f$ uniformly on $[a,b]$.

Apply WAT repeatedly with $\varepsilon_n = 1/n$ to get (p_n) s.t.

$$|p_n(x) - f(x)| < \frac{1}{n}$$

Then $p_n \to f$ uniformly.

Definition 6.7.1. A continuous function $\phi : [a,b] \to \mathbf{R}$ is polygonal if there is a partition

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

of $[a,b]$ such that ϕ is linear on each subinterval $[x_{i-1}, x_i]$, where $i = 1, \ldots, n$.

Theorem 6.7.2. *Let $f : [a,b] \to \mathbf{R}$ be continuous. Given $\epsilon > 0$, there exists a polygonal function ϕ satisfying*

$$|f(x) - \phi(x)| < \epsilon$$

for all $x \in [a,b]$.

Example 2. Prove Theorem 6.7.2.

f is continuous on $[a,b]$ and $[a,b]$ is compact $\Rightarrow$ f is uniformly continuous

$\Rightarrow$ Given $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $\forall$ $x, y \in [a,b]$, $|x-y| < \delta$ implies $|f(x) - f(y)| < \frac{\varepsilon}{2}$

Partition $[a,b]$ into uniform segments of length less than δ.

Define $\phi(x)$ at the endpoints of each segment to be $f(x)$ and linearly interpolate between end points.

For $x \in (a,b)$, let q be the largest end point less than x and r be the next endpoint.

Then $|x - \ | < \delta \Rightarrow |f(x) - \phi(q)| < \frac{\varepsilon}{2}$ and $|\phi(q) - \phi(r)| < \frac{\varepsilon}{2}$.

Since $\phi(x)$ lies between $\phi(q)$ and $\phi(r)$,

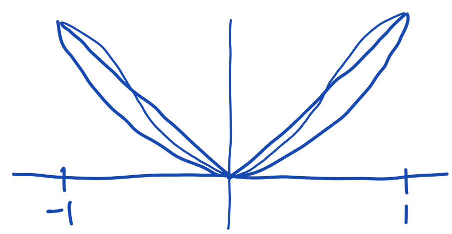
$$|f(x) - \phi(x)| \le |f(x) - \phi(q)| + |\phi(q) - \phi(x)| \le |f(x) - \phi(q)| + |\phi(q) - \phi(r)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Example 3. (a) Find the second degree polynomial $p(x) = q_0 + q_1 x + q_2 x^2$ that interpolates the three points $(-1, 1)$, $(0, 0)$ and $(1, 1)$ on the graph of $g(x) = |x|$. Sketch $g(x)$ and $p(x)$ over $[-1, 1]$ on the same set of axes.

(b) Find the fourth degree polynomial that interpolates $g(x) = |x|$ at the points $x = -1, -1/2, 0, 1/2$, and 1. Add a sketch of this polynomial to the graph from (a).

a) $p(0) = 0 \Rightarrow q_0 = 0$

$$\left.\begin{array}{l} \left.\begin{array}{l} p(-1) = 1 \Rightarrow -q_1 + q_2 = 1 \\ p(1) = 1 \Rightarrow q_1 + q_2 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} q_1 = 0 \\ q_2 = 1 \end{array} \end{array}\right\} \Rightarrow p(x) = x^2$$



b) $g(0) = 0$ and the polynomial h is even

$\Rightarrow h(x) = ax^2 + bx^4$

$$\left.\begin{array}{l} h(\frac{1}{2}) = \frac{1}{2} \Rightarrow \frac{1}{4}a + \frac{1}{16}b = \frac{1}{2} \\ h(1) = 1 \Rightarrow a + b = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} a = \frac{7}{3} \\ b = -\frac{4}{3} \end{array}\right\} \Rightarrow h(x) = \frac{7}{3}x^2 - \frac{4}{3}x^4$$

Example 4. Show that $f(x) = \sqrt{1-x}$ has Taylor series coefficients a_n where $a_0 = 1$ and

$$a_n = \frac{-1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for $n \geq 1$.

$$f^{(n)}(x) = -\frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n}(1-x)^{-\frac{2n-1}{2}}$$

$$a_n = \frac{f^{(n)}(0)}{n!} = -\frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} = \frac{-1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

Example 5. Use Example 4 of Section 6.6 to prove that

$$\sqrt{1-x} = \sum_{n=0}^{\infty} a_n x^n$$

for all $x \in (-1, 1)$.

Fix $x \in (-1,1)$. By the Cauchy formula,

$$E_N(x) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x = \frac{1}{N!}\left(\frac{-1\cdot 3\cdot 5\cdots(2N-1)}{2^{N+1}(1-c)^{N+1/2}}\right)(x-c)^N x = \left(\left(\frac{1}{2}\right)\frac{-1\cdot 3\cdot 5\cdots(2N-1)}{2\cdot 4\cdot 6\cdots(2N)}\right)\left(\frac{x-c}{1-c}\right)^N \frac{x}{\sqrt{1-c}}$$

$\left|\frac{x-c}{1-c}\right|$ is largest when $c=0 \Rightarrow \left|\frac{x-c}{1-c}\right| \le |x|$

$$\Rightarrow |E_N(x)| \le \frac{1}{2}|x|^N \frac{1}{\sqrt{1-|x|}} \longrightarrow 0 \text{ as } N \to \infty$$

Example 6. (a) Let

$$c_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}$$

for $n \ge 1$. Show $c_n < \frac{2}{\sqrt{2n+1}}$.

(b) Use (a) to show that $\sum_{n=0}^{\infty} a_n$ converges (absolutely, in fact) where a_n is the sequence of Taylor coefficients generated in Example 4.

(c) Carefully explain how this verifies that

$$\sqrt{1-x} = \sum_{n=0}^{\infty} a_n x^n$$

for all $x \in [-1, 1]$.

a) $n=1: c_1 = \frac{1}{2} < \frac{2}{\sqrt{3}}$

Assume true for $n=k$. Then $\frac{2k+1}{2k+2} < \sqrt{\frac{2k+1}{2k+3}} \Leftrightarrow 8k^3+20k^2+14k+3 < 8k^3+20k^2+16k+4 \Leftrightarrow k > -\frac{1}{2}$

So $c_{k+1} = c_k \frac{2k+1}{2k+2} < \frac{2}{\sqrt{2k+1}}\sqrt{\frac{2k+1}{2k+3}} = \frac{2}{\sqrt{2k+3}}$, and the result follows by induction

b) $|a_n| = \frac{c_n}{2n-1} < \frac{2}{(2n-1)\sqrt{2n+1}} \le \frac{2}{(2n-1)^{3/2}}$ and $\sum 1/n^{3/2}$ converges $\Rightarrow \sum a_n$ converges by comparison

c) $\sqrt{1-x}$ converges absolutely at 1 by (b) $\Rightarrow \sqrt{1-x}$ converges uniformly on $[-1,1]$

$\Rightarrow \sqrt{1-x}$ is continuous on $[-1,1]$

Also, $\sqrt{1-x} = \sum_{n=0}^{\infty} a_n x^n$ for $x \in (-1,1)$ by Ex 4, and this is also continuous

Thus $\lim_{x\to\pm 1}\sqrt{1-x} = \lim_{x\to\pm}\sum_{n=0}^{\infty} a_n x^n \Rightarrow \sqrt{1-x} = \sum_{n=0}^{\infty} a_n x^n \;\forall x \in [-1,1]$

Example 7. (a) Use the fact that $|a| = \sqrt{a^2}$ to prove that, given $\epsilon > 0$, there exists a polynomial $q(x)$ satisfying

$$||x| - q(x)| < \epsilon$$

for all $x \in [-1, 1]$.

(b) Generalize this conclusion to an arbitrary interval $[a, b]$.

a) Let $\varepsilon > 0$. $\sqrt{1-x} = \sum_{n=0}^{\infty} a_n x^n \quad \forall x \in [-1,1]$

$$\Rightarrow \exists\, S_N(x) = \sum_{n=0}^{N} a_n x^n \text{ s.t. } |\sqrt{1-y} - S_N(y)| < \varepsilon \quad \forall\, y \in [-1,1]$$

Let $y = 1 - x^2$. Then $x \in [-1,1] \Rightarrow y \in [0,1]$.

Since $|x| = \sqrt{x^2} = \sqrt{1-(1-x^2)} = \sqrt{1-y}$, we have $\left||x| - S_N(1-x^2)\right| < \varepsilon \quad \forall\, x \in [-1,1]$

Take $q(x) = S_N(1-x^2)$

b) Let $c = \max\{|a|, |b|\}$ and let $\varepsilon > 0$. Then by (a) $\exists$ a polynomial $p(x)$ s.t.

$$\left|\left|\frac{x}{c}\right| - p\left(\frac{x}{c}\right)\right| < \frac{\varepsilon}{c} \quad \forall\, x \in [-c, c]$$

$$\Rightarrow \left||x| - c \cdot p\left(\frac{x}{c}\right)\right| < \varepsilon \quad \forall\, x \in [-c, c]$$

Since $[a,b] \subseteq [-c,c]$, take $q(x) = c \cdot p\left(\frac{x}{c}\right)$

Example 8. (a) Fix $a \in [-1, 1]$ and sketch

$$h_a(x) = \frac{1}{2}(|x - a| + (x - a))$$

a)

over $[-1, 1]$. Note that h_a is polygonal and satisfies $h_a(x) = 0$ for all $x \in [-1, a]$.

(b) Explain why we know $h_a(x)$ can be uniformly approximated with a polynomial on $[-1, 1]$.

(c) Let ϕ be a polygonal function that is linear on each subinterval of the partition

$$-1 = a_0 < a_1 < a_2 < \cdots < a_n = 1.$$

Show there exist constants $b_0, b_1, \ldots, b_{n-1}$ so that

$$\phi(x) = \phi(-1) + b_0 h_{a_0}(x) + b_1 h_{a_1}(x) + \cdots + b_{n-1} h_{a_{n-1}}(x)$$

for all $x \in [-1, 1]$.

(d) Complete the proof of WAT for the interval $[-1, 1]$, and then generalize to an arbitrary interval $[a, b]$.

b) By Ex 7, given ε $\exists$ a polynomial $p(x)$ s.t. $\left|\left|\frac{x-a}{2}\right| - p\left(\frac{x-a}{2}\right)\right| < \varepsilon$ $\forall$ $\frac{x-a}{2} \in \left[\frac{-1-a}{2}, \frac{1-a}{2}\right]$

Then $\left|h_a(x) - \left(p\left(\frac{x-a}{2}\right) + \left(\frac{x-a}{2}\right)\right)\right| = \left|\left|\frac{x-a}{2}\right| - p\left(\frac{x-a}{2}\right)\right| < \varepsilon$ $\forall$ $x \in [-1,1]$.

Take $q(x) = p\left(\frac{x-a}{2}\right) + \left(\frac{x-a}{2}\right)$.

c) Set $b_0 = \frac{\phi(a_1) - \phi(a_0)}{a_1 - a_0}$, $b_n = \frac{\phi(a_{n+1}) - \phi(a_n)}{a_{n+1} - a_n} - b_{n-1}$ for $n \geq 1$

d) Fix $\varepsilon > 0$. f continuous on $[-1,1] \Rightarrow \exists$ polygonal $\phi(x)$ s.t. $|f(x) - \phi(x)| < \frac{\varepsilon}{2}$ $\forall x \in [-1,1]$

We also have $|\phi(x) - q(x)| < \frac{\varepsilon}{2}$ $\forall$ $x \in [-1,1]$ for some polynomial $q(x)$

$\Rightarrow |f(x) - q(x)| \leq |f(x) - \phi(x)| + |\phi(x) - q(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\forall$ $x \in [-1,1]$

Let $\ell(x) = mx + B$ be the linear function that maps $[-1,1]$ to $[a,b]$,

i.e., $m = \frac{b-a}{2}$ and $B = \frac{b+a}{2}$. f continuous on $[a,b] \Rightarrow f(\ell(x))$ is continuous on $[-1,1]$

$\Rightarrow \exists\, p(x)$ s.t. $|f(\ell(x)) - p(x)| < \varepsilon$ $\forall$ $x \in [-1,1]$.

Set $y = \ell(x)$. Then $|f(y) - p(\ell^{-1}(y))| < \varepsilon$ $\forall$ $y \in [a,b]$ where $p(\ell^{-1}(y))$ is a polynomial.

Example 9. (a) Find a counterexample which shows that WAT is not true if we replace the closed interval $[a, b]$ with the open interval (a, b).

(b) What happens if we replace $[a, b]$ with the closed set $[a, \infty)$? Does the theorem still hold?

a) Let $f(x) = 1/x$ on $(0,1)$. f is unbounded $\Rightarrow$ no polynomial can uniformly approximate f.

b) Let $g(x) = e^x$, which grows too fast to be uniformly approximated by any polynomial.

Or consider $\sin x$, which is bounded as $x \to \infty$, unlike any polynomial.

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Bibliography

[1] Stephen Abbott. *Understanding Analysis*. Springer, 2015. Print.