Calculus Notes

Calculus: Early Transcendentals 8th Edition
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Contents

Chapter 1

Functions and Models

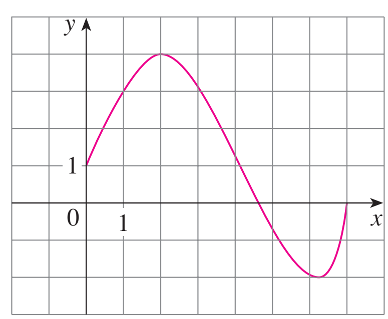
1.1 Four Ways to Represent a Function

Definition 1.1.1. A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E. The set D is called the domain of the function. The number $f(x)$ is the value of f at x. The set of all possible values of $f(x)$ as x varies throughout the domain is called the range. A symbol that represents a number in the domain of a function f is called an independent variable. A symbol that represents a number in the range of f is called a dependent variable.

Definition 1.1.2. If f is a function with domain D, then its graph is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}.$$

Example 1. The graph of a function f is shown in the figure.



(a) Find the values of $f(1)$ and $f(5)$.

(b) What are the domain and range of f?

Example 2. Sketch the graph and find the domain and range of each function.

(a) $f(x) = 2x - 1$

(b) $g(x) = x^2$

Example 3. If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\dfrac{f(a+h) - f(a)}{h}$.

Example 4. When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

Example 5. A rectangular storage container with an open top has a volume of 10 m^3. The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.

Example 6. Find the domain of each function.

(a) $f(x) = \sqrt{x+2}$

(b) $g(x) = \dfrac{1}{x^2 - x}$

Theorem 1.1.1 (Vertical Line Test)**.** *A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.*

Definition 1.1.3. Piecewise defined functions are defined by different formulas in different parts of their domains.

Example 7. A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq -1, \\ x^2 & \text{if } x > -1. \end{cases}$$

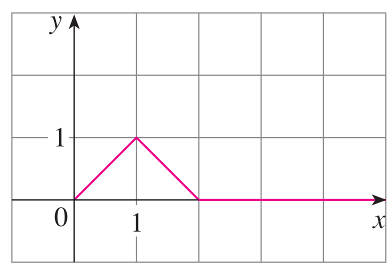
Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

Definition 1.1.4. The absolute value of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line.

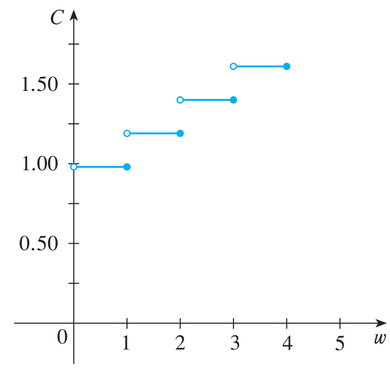
$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

Example 8. Sketch the graph of the absolute value function $f(x) = |x|$.

Example 9. Find a formula for the function f graphed in the figure.



Example 10. The cost $C(w)$ of mailing a large envelope with weight w is a piecewise defined function because, from the table of values representing the function,



w (ounces)	$C(w)$ (dollars)
$0 < w \leq 1$	0.98
$1 < w \leq 2$	1.19
$2 < w \leq 3$	1.40
$3 < w \leq 4$	1.61
$\vdots$	$\vdots$

we have

$$C(w) = \begin{cases} 0.98 & \text{if } 0 < w \leq 1, \\ 1.19 & \text{if } 1 < w \leq 2, \\ 1.40 & \text{if } 2 < w \leq 3, \\ 1.61 & \text{if } 3 < w \leq 4, \\ \vdots & \end{cases}$$

The graph is shown in the figure.

Remark 1. Functions similar to the one in the previous example are called step functions.

Definition 1.1.5. If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an even function.

Remark 2. The function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x).$$

Definition 1.1.6. If a function f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an odd function.

Remark 3. The function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

Example 11. Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$

(b) $g(x) = 1 - x^4$

(c) $h(x) = 2x - x^2$

Definition 1.1.7. A function f is called increasing on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I.$$

It is called decreasing on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I.$$

1.2 Mathematical Models

Definition 1.2.1. We say y is a linear function of x if the graph of the function is a line. The slope-intercept form of the equation of a line can be used to write a formula for the function as

$$y = f(x) = mx + b$$

where m is the slope of the line and b is the y-intercept.

Example 1. (a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature T (in °C) as a function of the height h (in kilometers), assuming that a linear model is appropriate.

(b) Draw the graph of the function in part (a). What does the slope represent?

(c) What is the temperature at a height of 2.5 km?

Definition 1.2.2. An empirical model is a model based entirely on collected data.

Example 2. The table lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2012. Use the data in the table to find a model for the carbon dioxide level.

Year	CO_2 level (in ppm)	Year	CO_2 level (in ppm)
1980	338.7	1998	366.5
1982	341.2	2000	369.4
1984	344.4	2002	373.2
1986	347.2	2004	377.5
1988	351.5	2006	381.9
1990	354.2	2008	385.6
1992	356.3	2010	389.9
1994	358.6	2012	393.8
1996	362.4		

Example 3. Use the linear model from the previous example to estimate the average CO_2 level for 1987 and to predict the level for the year 2020. According to this model, when will the CO_2 level exceed 420 parts per million?

Definition 1.2.3. A function P is called a polynomial if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the coefficients of the polynomial. If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n.

Remark 1. The function

$$P(x) = 2x^6 - x^4 + \frac{2}{5}x^3 + \sqrt{2}$$

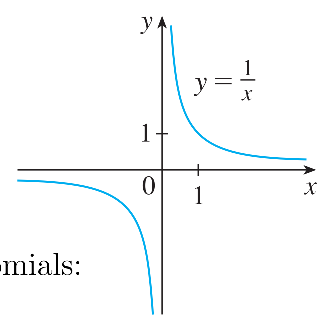
is a polynomial of degree 6.

Remark 2. A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function. A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a quadratic function. A polynomial of degree 3 is of the form $P(x) = ax^3 + bx^2 + cx + d$ and is called a cubic function.

Example 4. A ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground, and its height h above the ground is recorded at 1-second intervals in the table. Find a model to fit the data and use the model to predict the time at which the ball hits the ground.

Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

Definition 1.2.4. A function of the form $f(x) = x^a$, where a is a constant, is called a power function. If $a = n$, where n is a positive integer, $f(x) = x^n$ is a polynomial. If $a = 1/n$, where n is a positive integer, $f(x) = x^{1/n} = \sqrt[n]{x}$ is a root function. If $a = -1$, $f(x) = x^{-1} = 1/x$ is a reciprocal function, as shown in the figure.



Definition 1.2.5. A rational function f is a ratio of two polynomials:

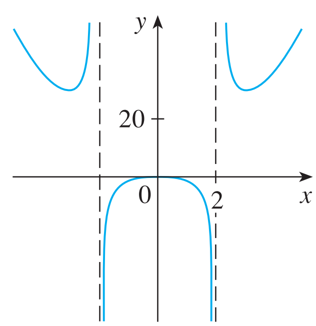
$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

Remark 3. The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$ and is graphed in the figure.



Definition 1.2.6. A function f is called an algebraic function if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials.

Remark 4. The functions

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

are algebraic.

Definition 1.2.7. Trigonometric functions are functions of an angle that relate the angles of a triangle to the lengths of its sides.

Remark 5. The sine, cosine, and tangent functions are the most familiar trigonometric functions. The convention in calculus is that radian measure is always used, unless otherwise indicated.

Remark 6. For all values of x, we have

$$-1 \leq \sin x \leq 1 \qquad -1 \leq \cos x \leq 1,$$

or equivalently,

$$|\sin x| \leq 1 \qquad |\cos x| \leq 1.$$

Also, the periodic nature of these functions implies that

$$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$$

for all values of x.

Example 5. What is the domain of the function $f(x) = \dfrac{1}{1 - 2\cos x}$?

Definition 1.2.8. Exponential functions are functions of the form $f(x) = b^x$, where the base b is a positive constant.

Definition 1.2.9. Logarithmic functions are functions of the form $f(x) = \log_b x$, where the base b is a positive constant.

Remark 7. Logarithmic functions are inverse functions of exponential functions.

Example 6. Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \dfrac{1+x}{1-\sqrt{x}}$

(d) $u(t) = 1 - t + 5t^4$

1.3 New Functions from Old Functions

Remark 1 (Vertical and Horizontal Shifts). Suppose $c > 0$. To obtain the graph of
$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Remark 2 (Vertical and Horizontal Stretching and Reflecting). Suppose $c > 1$. To obtain the graph of
$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Example 1. Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x-2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

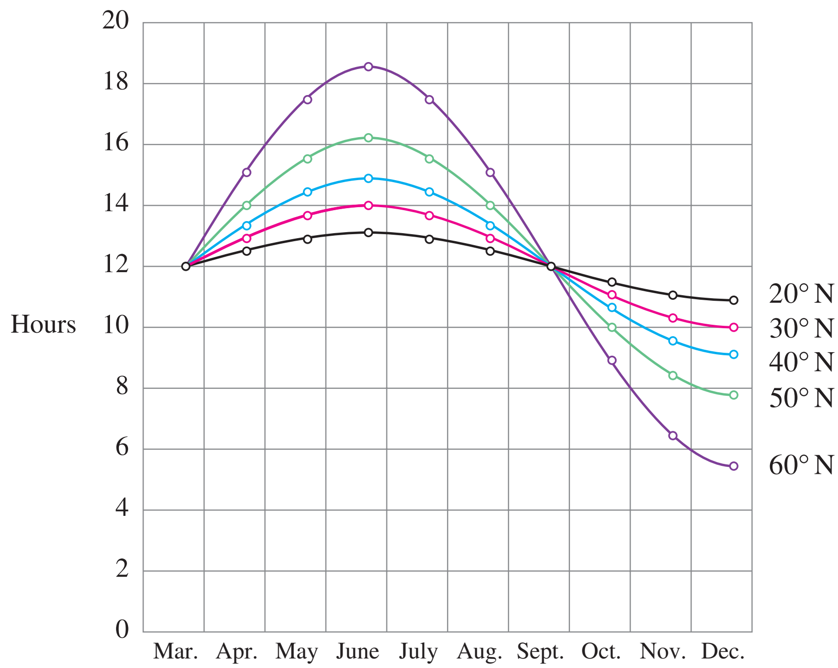
Example 2. Sketch the graph of the function $f(x) = x^2 + 6x + 10$.

Example 3. Sketch the graphs of the following functions.

(a) $y = \sin 2x$

(b) $y = 1 - \sin x$

Example 4. The figure shows graphs of the number of hours of daylight as functions of time of the year at several latitudes. Given that Philadelphia is located at approximately 40°N latitude, find a function that models the length of daylight at Philadelphia.



Example 5. Sketch the graph of the function $y = |x^2 - 1|$.

Definition 1.3.1. The sum and difference functions are defined by

$$(f+g)(x) = f(x) + g(x) \qquad (f-g)(x) = f(x) - g(x).$$

Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0.$$

Definition 1.3.2. Given two functions f and g, the composite function $f \circ g$ (also called the composition of f and g) is defined by

$$(f \circ g)(x) = f(g(x)).$$

Example 6. If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

Example 7. If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$, find each of the following functions and their domains.

(a) $f \circ g$

(b) $g \circ f$

(c) $f \circ f$

(d) $g \circ g$

Example 8. Find $f \circ g \circ h$ if $f(x) = x/(x+1)$, $g(x) = x^{10}$, and $h(x) = x+3$.

Example 9. Given $F(x) = \cos^2(x+9)$, find functions f, g, and h such that $F = f \circ g \circ h$.

1.4 Exponential Functions

Theorem 1.4.1 (Laws of Exponents)**.** *If a and b are positive numbers and x and y are any real numbers, then*

1. $b^{x+y} = b^x b^y$ 2. $b^{x-y} = \dfrac{b^x}{b^y}$ 3. $(b^x)^y = b^{xy}$ 4. $(ab)^x = a^x b^x$

Example 1. Sketch the graph of the function $y = 3 - 2^x$ and determine its domain and range.

Example 2. Use a graphing calculator to compare the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$. Which function grows more quickly when x is large?

Example 3. The half-life of strontium-90, ^{90}Sr, is 25 years. This means that half of any given quantity of ^{90}Sr will disintegrate in 25 years.

(a) If a sample of ^{90}Sr has a mass of 24 mg, find an expression for the mass $m(t)$ that remains after t years.

(b) Find the mass remaining after 40 years, correct to the nearest milligram.

(c) Use a graphing calculator to graph $m(t)$ and use the graph to estimate the time required for the mass to be reduced to 5 mg.

Definition 1.4.1. We call the function $f(x) = e^x$ the natural exponential function where e is the value of b in $y = b^x$ resulting in a tangent line at $(0, 1)$ with slope 1.

Example 4. Graph the function $y = \frac{1}{2}e^{-x} - 1$ and state the domain and range.

Example 5. Use a graphing device to find the values of x for which $e^x > 1,000,000$.

1.5 Inverse Functions and Logarithms

Definition 1.5.1. A function is a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2.$$

Theorem 1.5.1 (Horizontal Line Test)**.** *A function is one-to-one if and only if no horizontal line intersects its graph more than once.*

Example 1. Is the function $f(x) = x^3$ one-to-one?

Example 2. Is the function $g(x) = x^2$ one-to-one?

Definition 1.5.2. Let f be a one-to-one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

for any y in B.

Example 3. If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

Remark 1. The letter x is traditionally used as the independent variable, so when we concentrate on f^{-1} we usually reverse the roles of x and y to get

$$f^{-1}(x) = y \Leftrightarrow f(y) = x.$$

By substituting for x and y, we get the following cancellation equations:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$
$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Example 4. Find the inverse function of $f(x) = x^3 + 2$.

Remark 2. The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Example 5. Sketch the graphs of $f(x) = \sqrt{-1-x}$ and its inverse function using the same coordinate axes.

Definition 1.5.3. The logarithmic function with base b, denoted by $\log_b$, is the inverse function of the exponential function $f(x) = b^x$ with $b > 0$ and $b \neq 1$, i.e.,

$$\log_b x = y \Leftrightarrow b^y = x.$$

Remark 3. By the cancellation equations,

$$\begin{aligned} \log_b(b^x) &= x \quad \text{for every } x \in \mathbb{R} \\ b^{\log_b x} &= x \quad \text{for every } x > 0. \end{aligned}$$

Theorem 1.5.2 (Laws of Logarithms)**.** *If x and y are positive numbers, then*

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b(x^r) = r \log_b x$ *(where r is any real number)*

Example 6. Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

Definition 1.5.4. The logarithm with base e is called the natural logarithm and is denoted by

$$\log_e x = \ln x.$$

Example 7. Find x if $\ln x = 5$.

Example 8. Solve the equation $e^{5-3x} = 10$.

Example 9. Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

Theorem 1.5.3 (Change of Base Formula)**.** *For any positive number b ($b \neq 1$), we have*

$$\log_b x = \frac{\ln x}{\ln b}.$$

Proof. Let $y = \log_b x$. Then

$$\begin{aligned} b^y &= x \\ y \ln b &= \ln x \\ y &= \frac{\ln x}{\ln b}. \end{aligned}$$

□

Example 10. Evaluate $\log_8 5$ correct to six decimal places.

Example 11. Sketch the graph of the function $y = \ln(x-2) - 1$.

Definition 1.5.5. The <u>inverse sine function</u> or <u>arcsine function</u>, denoted by $\sin^{-1}$, is the inverse of the sine function on the restricted domain $[-\pi/2, \pi/2]$.

Remark 4. By the cancellation equations,

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
$$\sin(\sin^{-1} x) = x \quad \text{for } -1 \leq x \leq 1.$$

Example 12. Evaluate (a) $\sin^{-1}(\frac{1}{2})$ and (b) $\tan\left(\arcsin \frac{1}{3}\right)$.

Definition 1.5.6. The <u>inverse cosine function</u> or <u>arccosine function</u>, denoted by $\cos^{-1}$, is the inverse of the cosine function on the restricted domain $[0, \pi]$.

Remark 5. By the cancellation equations,

$$\cos^{-1}(\cos x) = x \quad \text{for } 0 \leq x \leq \pi$$
$$\cos(\cos^{-1} x) = x \quad \text{for } -1 \leq x \leq 1.$$

Definition 1.5.7. The <u>inverse tangent function</u> or <u>arctangent function</u>, denoted by $\tan^{-1}$, is the inverse of the tangent function on the restricted domain $[-\pi/2, \pi/2]$.

Example 13. Simplify the expression $\cos(\tan^{-1} x)$.

Remark 6. The remaining inverse trigonometric functions are

$$\begin{aligned}
y = \csc^{-1} x \quad (|x| \geq 1) \quad &\iff \quad \csc y = x \quad \text{and} \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2] \\
y = \sec^{-1} x \quad (|x| \geq 1) \quad &\iff \quad \sec y = x \quad \text{and} \quad y \in [0, \pi/2) \cup [\pi, 3\pi/2) \\
y = \cot^{-1} x \quad (|x| \in \mathbb{R}) \quad &\iff \quad \cot y = x \quad \text{and} \quad y \in (0, \pi).
\end{aligned}$$

Chapter 2

Limits and Derivatives

2.1 The Tangent and Velocity Problems

Remark 1. A tangent to a curve is a line that that touches the curve. A secant is a line that cuts a curve more than once.

Example 1. Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Example 2. The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The data in the table describe the charge Q remaining on the capacitor (measured in microcoulombs) at time t (measured in seconds after the flash goes off). Use the data to draw the graph of this function and estimate the slope of the tangent line at the point where $t = 0.04$. [*Note:* The slope of the tangent line represents the electric current flowing from the capacitor to the flash bulb (measured in microamperes).]

t	Q
0.00	100.0
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

Example 3. Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds. [If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law that the distance fallen by any freely falling body is proportional to the square of the time it has been falling is expressed by the equation $s(t) = 4.9t^2$.]

2.2 The Limit of a Function

Definition 2.2.1. Suppose $f(x)$ is defined when x is near the number a. Then we write

$$\lim_{x \to a} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a but not equal to a.

Example 1. Guess the value of $\lim_{x \to 1} \dfrac{x-1}{x^2-1}$.

Example 2. Estimate the value of $\lim_{t \to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$.

Example 3. Guess the value of $\lim_{x \to 0} \dfrac{\sin x}{x}$.

Example 4. Investigate $\lim_{x \to 0} \sin \frac{\pi}{x}$.

Example 5. Find $\lim_{x \to 0} \left(x^3 + \frac{\cos 5x}{10,000} \right)$.

Definition 2.2.2. We write

$$\lim_{x \to a^-} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x less than a. Similarly, if we require that x be greater than a, we write

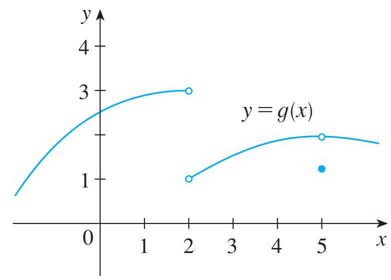
$$\lim_{x \to a^+} f(x) = L.$$

Example 6. Investigate the limit as t approaches 0 of the Heaviside function H, defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

Remark 1. $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

Example 7. Use the graph of g to state the values (if they exist) of the following:



(a) $\lim_{x\to 2^-} g(x)$ (b) $\lim_{x\to 2^+} g(x)$

(c) $\lim_{x\to 2} g(x)$ (d) $\lim_{x\to 5^-} g(x)$

(e) $\lim_{x\to 5^+} g(x)$ (f) $\lim_{x\to 5} g(x)$

Definition 2.2.3. Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a. Similarly,

$$\lim_{x\to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

Example 8. Find $\lim_{x\to 0} \frac{1}{x^2}$ if it exists.

Definition 2.2.4. The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$
$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Example 9. Find $\displaystyle\lim_{x \to 3^+} \frac{2x}{x-3}$ and $\displaystyle\lim_{x \to 3^-} \frac{2x}{x-3}$.

Example 10. Find the vertical asymptotes of $f(x) = \tan x$.

2.3 Calculating Limits Using the Limit Laws

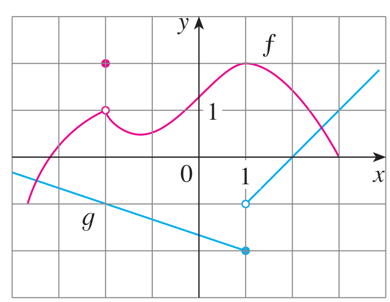
Theorem 2.3.1 (Limit Laws)**.** *Suppose that c is a constant and the limits*

$$\lim_{x\to a} f(x) \qquad \textit{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0$

Example 1. Use the Limit Laws and the graphs of f and g to evaluate the following limits, if they exist.



(a) $\lim_{x\to -2}[f(x)+5g(x)]$

(b) $\lim_{x\to 1}[f(x)g(x)]$

(c) $\lim_{x\to 2} \dfrac{f(x)}{g(x)}$

Theorem 2.3.2 (Power and Root Laws)**.** *By repeatedly applying the Product Law and using some basic intuition we obtain the following:*

6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ *where n is a positive integer*
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ *where n is a positive integer*
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ *where n is a positive integer*
(If n is even, we assume that $a > 0$.)
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ *where n is a positive integer*
$\left[\textit{If } n \textit{ is even, we assume that } \lim_{x\to a} f(x) > 0.\right]$

Example 2. Evaluate the following limits and justify each step.

(a) $\lim_{x\to 5}(2x^2 - 3x + 4)$

(b) $\lim_{x\to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

Theorem 2.3.3 (Direct Substitution Property)**.** *If f is a polynomial or a rational function and a is in the domain of f, then*

$$\lim_{x \to a} f(x) = f(a).$$

Example 3. Find $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$.

Remark 1. If $f(x) = g(x)$ when $x \neq a$, then $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$, provided the limits exist.

Example 4. Find $\lim\limits_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} x + 1 & \text{if } x \neq 1, \\ \pi & \text{if } x = 1. \end{cases}$$

Example 5. Evaluate $\lim\limits_{h \to 0} \dfrac{(3 + h)^2 - 9}{h}$.

Example 6. Find $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$.

Example 7. Show that $\lim_{x \to 0} |x| = 0$.

Example 8. Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Example 9. If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4, \\ 8-2x & \text{if } x < 4. \end{cases}$$

determine whether $\lim_{x \to 4} f(x)$ exists.

Example 10. The greatest integer function is defined by $[\![x]\!]$ = the largest integer that is less than or equal to x. (For instance, $[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}]\!] = 1$, $[\![-\frac{1}{2}]\!] = -1$.) Show that $\lim_{x \to 3} [\![x]\!]$ does not exist.

Theorem 2.3.4. *If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then*

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

Theorem 2.3.5 (The Squeeze Theorem)**.** *If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L.$$

Example 11. Show that $\lim_{x \to 0} x^2 \sin \dfrac{1}{x} = 0$.

2.4 The Precise Definition of a Limit

Definition 2.4.1. Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Example 1. Use a graph to find a number δ such that if x is within δ of 1, then $f(x) = x^3 - 5x + 6$ is within 0.2 of 2.

Example 2. Prove that $\lim_{x \to 3}(4x - 5) = 7$.

Definition 2.4.2.

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Similarly,

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Example 3. Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Example 4. Prove that $\lim_{x \to 3} x^2 = 9$.

Definition 2.4.3. Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) > M.$$

Similarly,

$$\lim_{x \to a} f(x) = -\infty$$

means that for every negative number N there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) < N.$$

Example 5. Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

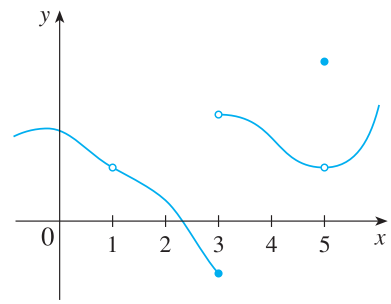
2.5 Continuity

Definition 2.5.1. A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a).$$

We say that f is discontinuous at a (or f has a discontinuity at a) if f is not continuous at a.

Example 1. Use the graph of the function f to determine the numbers at which f is discontinuous.



Example 2. Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

Definition 2.5.2. A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

Example 3. In which direction(s) is the function $f(x) = [\![x]\!]$ continuous?

Definition 2.5.3. A function f is continuous on an interval if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.)

Example 4. Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

Theorem 2.5.1. *If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:*

1. $f + g$	*2.* $f - g$	*3.* cf
4. fg	*5.* $\dfrac{f}{g}$ *if* $g(a) \neq 0$	

Proof. All of these results follow from the Limit Laws. For example, $f + g$ is continuous at a because

$$\begin{aligned}\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)]\\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x)\\ &= f(a) + g(a)\\ &= (f+g)(a).\end{aligned}$$

□

Theorem 2.5.2. *(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.*

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Proof. (a) Let

$$P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

be a polynomial where $c_0, c_1, \ldots, c_n$ are constants. Then

$$\lim_{x\to a} x^m = a^m \qquad m = 1, 2, \ldots, n$$

implies that the function $f(x) = x^m$ is continuous. Since

$$\lim_{x\to a} c_0 = c_0,$$

the constant function is continuous as well, and therefore the product function $g(x) = cx^m$ is continuous. Since P is a sum of functions of this form, it is continuous as well.

(b) Rational functions are quotients of polynomials, i.e.,

$$f(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials. Thus the above result implies that they are continuous on their domains. □

Example 5. Find $\lim_{x\to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$.

Theorem 2.5.3. *The following types of functions are continuous at every number in their domains:*

- *polynomials*
- *rational functions*
- *root functions*
- *trigonometric functions*
- *inverse trigonometric functions*
- *exponential functions*
- *logarithmic functions*

Example 6. Where is the function $f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?

Example 7. Evaluate $\lim_{x\to\pi} \frac{\sin x}{2 + \cos x}$.

Theorem 2.5.4. *If f is continuous at b and $\lim\limits_{x\to a} g(x) = b$, then $\lim\limits_{x\to a} f(g(x)) = f(b)$, i.e.,*

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

Proof. Let $\varepsilon > 0$. Since f is continuous at b, we have $\lim_{y\to b} f(y) = f(b)$ and so there exists $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |y - b| < \delta_1 \qquad \text{then} \qquad |f(y) - f(b)| < \varepsilon.$$

Since $\lim_{x\to a} g(x) = b$, there exists $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |g(x) - b| < \delta_1.$$

By letting $y = g(x)$ in the first statement, we get that $0 < |x - a| < \delta$ implies that $\big|f(g(x)) - f(b)\big| < \varepsilon$, i.e., $\lim_{x\to a} f(g(x)) = f(b)$. □

Example 8. Evaluate $\lim\limits_{x\to 1} \arcsin\left(\dfrac{1-\sqrt{x}}{1-x}\right)$.

Theorem 2.5.5. *If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.*

Proof. Since g is continuous at a, we have

$$\lim_{x\to a} g(x) = g(a).$$

Since f is continuous at $g(a)$, we have

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(g(a)),$$

which means $f \circ g$ is continuous. □

Example 9. Where are the following functions continuous?

(a) $h(x) = \sin(x^2)$

(b) $F(x) = \ln(1 + \cos x)$

Theorem 2.5.6 (Intermediate Value Theorem)**.** *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.*

Example 10. Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

2.6 Limits at Infinity

Definition 2.6.1. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Definition 2.6.2. Let f be a function defined on some interval $(-\infty, a)$. Then

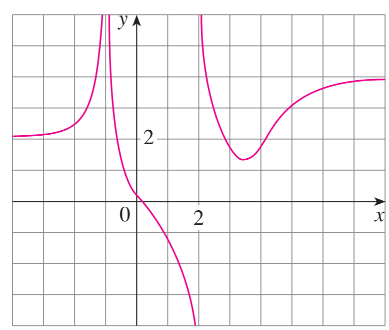
$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

Definition 2.6.3. The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L.$$

Example 1. Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown.



Example 2. Find $\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$.

Theorem 2.6.1. *If $r > 0$ is a rational number, then*

$$\lim_{x\to\infty} \frac{1}{x^r} = 0.$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0.$$

Proof. By extending the limit laws to limits at infinity we get

$$\lim_{x\to\infty} \frac{1}{x^r} = \lim_{x\to\infty} \left[\frac{1}{x}\right]^r = \left[\lim_{x\to\infty} \frac{1}{x}\right]^r = 0^r = 0$$
$$\lim_{x\to-\infty} \frac{1}{x^r} = \lim_{x\to-\infty} \left[\frac{1}{x}\right]^r = \left[\lim_{x\to-\infty} \frac{1}{x}\right]^r = 0^r = 0.$$

□

Example 3. Evaluate

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}.$$

Example 4. Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}.$$

Example 5. Compute $\lim_{x\to\infty}(\sqrt{x^2+1}-x)$.

Example 6. Evaluate $\lim_{x\to 2^+}\arctan\left(\frac{1}{x-2}\right)$.

Example 7. Evaluate $\lim_{x\to 0^-} e^{1/x}$.

Example 8. Evaluate $\lim_{x\to\infty}\sin x$.

Example 9. Find $\lim_{x \to \infty} x^3$ and $\lim_{x \to -\infty} x^3$.

Example 10. Find $\lim_{x \to \infty} (x^2 - x)$.

Example 11. Find $\lim_{x \to \infty} \frac{x^2 + x}{3 - x}$.

Example 12. Sketch the graph of $y = (x-2)^4(x+1)^3(x-1)$ by finding its intercepts and its limits as $x \to \infty$ and as $x \to -\infty$.

Definition 2.6.4. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Definition 2.6.5. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x < N \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Example 13. Use a graph to find a number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1.$$

Example 14. Prove that $\displaystyle\lim_{x \to \infty} \frac{1}{x} = 0$.

Definition 2.6.6. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad f(x) > M.$$

Definition 2.6.7. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = -\infty$$

means that for every negative number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad f(x) < M.$$

Definition 2.6.8. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = \infty$$

means that for every positive number M there is a corresponding negative number N such that

$$\text{if} \quad x < N \qquad \text{then} \qquad f(x) > M.$$

Definition 2.6.9. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = -\infty$$

means that for every negative number M there is a corresponding negative number N such that

$$\text{if} \quad x < N \qquad \text{then} \qquad f(x) < M.$$

2.7 Derivatives and Rates of Change

Definition 2.7.1. The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Example 1. Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Example 2. Use the alternative expression for the slope of a tangent line

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

to find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

Definition 2.7.2. A function f describing the motion of an object along a straight line is called a position function and has velocity

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

at time $t = a$.

Example 3. Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground. Recall that the distance (in meters) fallen after t seconds is $4.9t^2$.
(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Definition 2.7.3. The derivative of a function f at a number a, denoted by $f'(a)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

or equivalently

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

Example 4. Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

Example 5. Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

Definition 2.7.4. Suppose y is a quantity that depends on another quantity x. Then y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the increment of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1).$$

The average rate of change of y with respect x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

and the instantaneous rate of change of y with respect to x is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x).$$

Example 6. A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is $C = f(x)$ dollars.
(a) What is the meaning of the derivative of $f'(x)$? What are its units?

(b) In practical terms, what does it mean to say that $f'(1000) = 9$?

(c) Which do you think is greater, $f'(50)$ or $f'(500)$? What about $f'(5000)$?

Example 7. Let $D(t)$ be the US national debt at time t. The table gives approximate values of this function by providing end of year estimates, in billions of dollars, from 1985 to 2010. Interpret and estimate the value of $D'(2000)$.

t	$D(t)$
1985	1945.9
1990	3364.8
1995	4988.7
2000	5662.2
2005	8170.4
2010	$14,025.2$

Source: US Dept. of the Treasury

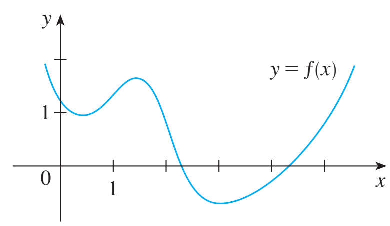
2.8 The Derivative as a Function

Definition 2.8.1. The derivative of a function f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if this limit exists.

Example 1. The graph of a function f is given. Use it to sketch the graph of the derivative f'.



Example 2. (a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.

(b) Illustrate this formula by comparing the graphs of f and f'.

Example 3. If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

Example 4. Find f' if $f(x) = \dfrac{1-x}{2+x}$.

Definition 2.8.2. The symbols D and d/dx are called differentiation operators and are used as follows:

$$f'(x) = y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x).$$

For fixed a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_{x=a}$$

Definition 2.8.3. A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Example 5. Where is the function $f(x) = |x|$ differentiable?

Theorem 2.8.1. *If f is differentiable at a, then f is continuous at a.*

Proof. If f is differentiable at a, we have

$$\begin{aligned}\lim_{x\to a}[f(x)-f(a)] &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}(x-a)\\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a)\\ &= f'(a)\cdot 0 = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}[f(a)+(f(x)-f(a))]\\ &= \lim_{x\to a} f(a)+\lim_{x\to a}[f(x)-f(a)]\\ &= f(a)+0 = f(a).\end{aligned}$$

□

Definition 2.8.4. If the derivative f' of a function f has a derivative of its own we call it the second derivative of f and denote it by

$$(f')' = f'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Example 6. If $f(x) = x^3 - x$, find and interpret $f''(x)$.

Definition 2.8.5. The instantaneous rate of change of velocity with respect to time is called the acceleration $a(t)$ of an object. It is the derivative of the velocity function, and therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t).$$

Definition 2.8.6. The third derivative f''' is the derivative of the second derivative, denoted by

$$(f'')' = f'''.$$

Definition 2.8.7. The instantaneous rate of change of acceleration with respect to time is called the jerk $j(t)$ of an object. It is the derivative of the acceleration function, and therefore the third derivative of the position function:

$$j(t) = a'(t) = v''(t) = s'''(t).$$

Definition 2.8.8. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Example 7. If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.

Chapter 3

Differentiation Rules

3.1 Derivatives of Polynomials and Exponentials

Theorem 3.1.1. *The derivative of a constant function* $f(x) = c$ *is* 0*, i.e.,*

$$\frac{d}{dx}(c) = 0.$$

Proof.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0. \qquad \square$$

Theorem 3.1.2.

$$\frac{d}{dx}(x) = 1 \qquad \frac{d}{dx}(x^2) = 2x \qquad \frac{d}{dx}(x^3) = 3x^2 \qquad \frac{d}{dx}(x^4) = 4x^3$$

Proof. All of these follow directly from the definition of the derivative, as above. □

Theorem 3.1.3 (The Power Rule)**.** *If* n *is a positive integer, then*

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof. Since

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}),$$

we have

$$\begin{aligned}
f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{x^n-a^n}{x-a} \\
&= \lim_{x\to a}(x^{n-1}+x^{n-2}a+\cdots+xa^{n-2}+a^{n-1}) \\
&= a^{n-1}+a^{n-2}a+\cdots+aa^{n-2}+a^{n-1} \\
&= \underbrace{a^{n-1}+a^{n-1}+\cdots+a^{n-1}+a^{n-1}}_{n} \\
&= na^{n-1}.
\end{aligned}$$

□

Example 1. Find the derivative of each of the following:
(a) $f(x) = x^6$

(b) $y = x^{1000}$

(c) $y = t^4$

(d) $f(r) = r^3$

Theorem 3.1.4 (The Power Rule (General Version))**.** *If n is any real number, then*

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Example 2. Differentiate:
(a) $f(x) = \dfrac{1}{x^2}$

(b) $y = \sqrt[3]{x^2}$

Definition 3.1.1. The normal line to a curve C at a point P is the line through P that is perpendicular to the tangent line at P.

Example 3. Find equations of the tangent line and normal line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$.

Theorem 3.1.5 (The Constant Multiple Rule)**.** *If c is a constant and f is a differentiable function, then*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x).$$

Proof. Let $g(x) = cf(x)$. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} \\ &= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right] \\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= cf'(x). \end{aligned}$$

□

Example 4. Find:

(a) $\dfrac{d}{dx}(3x^4)$

(b) $\dfrac{d}{dx}(-x)$

Theorem 3.1.6 (The Sum Rule)**.** *If f and g are both differentiable, then*

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Proof. Let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\ &= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right] \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x). \end{aligned}$$

□

Theorem 3.1.7 (The Difference Rule)**.** *If f and g are both differentiable, then*

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

Example 5. Find $\dfrac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$.

Example 6. Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

Example 7. The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

Definition 3.1.2. e is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

Theorem 3.1.8. $\frac{d}{dx}(e^x) = e^x$.

Example 8. If $f(x) = e^x - x$, find f' and f''.

Example 9. At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

3.2 The Product and Quotient Rules

Theorem 3.2.1 (The Product Rule)**.** *If f and g are both differentiable, then*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

Proof. By the definition of the derivative on the product,

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)[g(x+h) - g(x)]}{h} + \lim_{h\to 0}\frac{g(x)[f(x+h) - f(x)]}{h} \\
&= \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\
&= f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]. \qquad \square
\end{aligned}$$

Example 1. (a) If $f(x) = xe^x$, find $f'(x)$.

(b) Find the nth derivative, $f^{(n)}(x)$.

Example 2. Differentiate the function $f(t) = \sqrt{t}(a + bt)$.

Example 3. If $f(x) = \sqrt{x}g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

Theorem 3.2.2 (The Quotient Rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Proof. Similar to the Product Rule, except we add and subtract $f(x)g(x)$ in the numerator when applying the definition of the derivative. □

Example 4. Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Find y'.

Example 5. Find an equation of the tangent line to the curve $y = e^x/(1+x^2)$ at the point $(1, \frac{1}{2}e)$.

3.3 Derivatives of Trigonometric Functions

Theorem 3.3.1. *The derivative of the sine function is the cosine function, i.e.,*

$$\frac{d}{dx}(\sin x) = \cos x.$$

Example 1. Differentiate $y = x^2 \sin x$.

Theorem 3.3.2. *The derivative of the cosine function is the negative sine function, i.e.,*

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Theorem 3.3.3. *The derivative of the tangent function is the square of the secant function, i.e.,*

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

Proof. By the Quotient Rule,

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x.
\end{aligned}$$

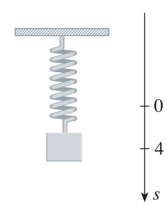
□

Theorem 3.3.4. *The derivatives of the trigonometric functions are*

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x.$$

Example 2. Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

Example 3. An object at the end of a vertical spring is stretched to 4 cm beyond its reset position and released at time $t = 0$. (See the figure and note that the downward direction is positive.) Its position at time t is



$$s = f(t) = 4 \cos t.$$

Find the velocity and acceleration at time t and use them to analyze the motion of the object.

Example 4. Find the 27th derivative of $\cos x$.

Example 5. Find $\lim\limits_{x\to 0} \dfrac{\sin 7x}{4x}$.

Example 6. Calculate $\lim\limits_{x\to 0} x \cot x$.

3.4 The Chain Rule

Theorem 3.4.1 (The Chain Rule)**.** *If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product*

$$F'(x) = f'(g(x)) \cdot g'(x).$$

Example 1. Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$.

Example 2. Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

Theorem 3.4.2 (The Power Rule Combined with the Chain Rule)**.** *If n is any real number and $u = g(x)$ is differentiable, then*

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

Example 3. Differentiate $y = (x^3 - 1)^{100}$.

Example 4. Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

Example 5. Find the derivative of the function

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9.$$

Example 6. Differentiate $y = (2x + 1)^5 (x^3 - x + 1)^4$.

Example 7. Differentiate $y = e^{\sin x}$.

Theorem 3.4.3. *The derivative of the exponential function is*

$$\frac{d}{dx}(b^x) = b^x \ln b.$$

Proof. Since

$$b^x = (e^{\ln b})^x = e^{(\ln b)x},$$

the Chain Rule gives

$$\begin{aligned} \frac{d}{dx}(b^x) &= \frac{d}{dx}(e^{(\ln b)x}) \\ &= e^{(\ln b)x} \frac{d}{dx}(\ln b)x \\ &= e^{(\ln b)x} \cdot \ln b \\ &= b^x \ln b. \end{aligned}$$

□

Example 8. Find $\dfrac{d}{dx}(2^x)$.

Example 9. Find $f'(x)$ if $f(x) = \sin(\cos(\tan x))$.

Example 10. Differentiate $y = e^{\sec 3\theta}$.

3.5 Implicit Differentiation

Definition 3.5.1. Implicit differentiation is the method of differentiation both sides of an equation with respect to x, and then solving the equation for y' when $y = f(x)$.

Example 1. (a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

Example 2. (a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

(c) At what point in the first quadrant is the tangent line horizontal?

Example 3. Find y' if $\sin(x + y) = y^2 \cos x$.

Example 4. Find y'' if $x^4 + y^4 = 16$.

Theorem 3.5.1. *The derivative of the arcsine function is*

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

Proof. Since $y = \sin^{-1} x$ means $\sin y = x$ and $-\pi/2 \leq y \leq \pi/2$, we have $\cos y \geq 0$. Thus we can differentiate to obtain

$$\begin{aligned} \sin y &= x \\ \cos y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y} \\ &= \frac{1}{\sqrt{1-\sin^2 y}} \\ &= \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

□

Theorem 3.5.2. *The derivative of the arctangent function is*

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

Proof. If $y = \tan^{-1} x$, then $\tan y = x$. Differentiating then gives us

$$\begin{aligned} \tan y &= x \\ \sec^2 y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} \\ &= \frac{1}{1+\tan^2 y} \\ &= \frac{1}{1+x^2}. \end{aligned}$$

□

Example 5. Differentiate

(a) $y = \dfrac{1}{\sin^{-1} x}$

(b) $f(x) = x \arctan \sqrt{x}$.

Theorem 3.5.3. *The derivatives of the Inverse Trigonometric Functions are*

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}.$$

Theorem 3.5.4. *Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Then f^{-1} has derivative*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.

Proof. Since $(f \circ f^{-1})(x) = x$, we have, by the chain rule,

$$\begin{aligned}(f \circ f^{-1})(x) &= x \\ (f \circ f^{-1})'(x) &= 1 \\ f'(f^{-1}(x))(f^{-1})'(x) &= 1 \\ (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))}.\end{aligned}$$

□

Example 6. If $f(4) = 5$ and $f'(4) = \frac{2}{3}$, find $(f^{-1})'(5)$.

3.6 Derivatives of Logarithmic Functions

Theorem 3.6.1. *The derivative of the logarithm function is*

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}.$$

Proof. Let $y = \log_b x$. Then $b^y = x$, so by differentiating we get

$$\begin{aligned} b^y &= x \\ b^y(\ln b)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{b^y \ln b} \\ &= \frac{1}{x \ln b}. \end{aligned}$$

□

Theorem 3.6.2. *The derivative of the natural logarithm is*

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Example 1. Differentiate $y = \ln(x^3 + 1)$.

Example 2. Find $\dfrac{d}{dx}\ln(\sin x)$.

Example 3. Differentiate $f(x) = \sqrt{\ln x}$.

Example 4. Differentiate $f(x) = \log_{10}(2 + \sin x)$.

Example 5. Find $\dfrac{d}{dx} \ln \dfrac{x+1}{\sqrt{x-2}}$.

Example 6. Find $f'(x)$ if $f(x) = \ln |x|$.

Definition 3.6.1. Logarithmic differentiation is the method of calculating derivatives of functions by taking logarithms, differentiating implicitly, and then solving the resulting equation for the derivative.

Example 7. Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

Theorem 3.6.3 (The Power Rule)**.** *If n is any real number and $f(x) = x^n$, then*

$$f'(x) = nx^{n-1}.$$

Proof. Let $y = x^n$. By logarithmic differentiation we get

$$\begin{aligned} y &= x^n \\ \ln|y| &= \ln|x|^n \\ &= n\ln|x| \qquad x \neq 0 \\ \frac{y'}{y} &= \frac{n}{x} \\ y' &= n\frac{y}{x} \\ &= n\frac{x^n}{x} \\ &= nx^{n-1}. \end{aligned}$$

□

Example 8. Differentiate $y = x^{\sqrt{x}}$.

Theorem 3.6.4. *The number e can be defined as the limit*

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Proof. If $f(x) = \ln x$, then $f'(1) = 1$, so

$$\begin{aligned} f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{x\to 0}\frac{f(1+x)-f(1)}{x} \\ &= \lim_{x\to 0}\frac{\ln(1+x)-\ln 1}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x) \\ &= \lim_{x\to 0}\ln(1+x)^{1/x} = 1. \end{aligned}$$

Thus

$$e = e^1 = e^{\left(\lim_{x\to 0}\ln(1+x)^{1/x}\right)} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}} = \lim_{x\to 0}(1+x)^{1/x}.$$

Then if we let $n = 1/x$, $n\to\infty$ as $x\to 0^+$, so we are done. $\square$

3.7 Rates of Change in the Sciences

Example 1. The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters.

(a) Find the velocity at time t.

(b) What is the velocity after 2 s? After 4 s?

(c) When is the particle at rest?

(d) When is the particle moving forward (that is, in the positive direction)?

(e) Draw a diagram to represent the motion of the particle.

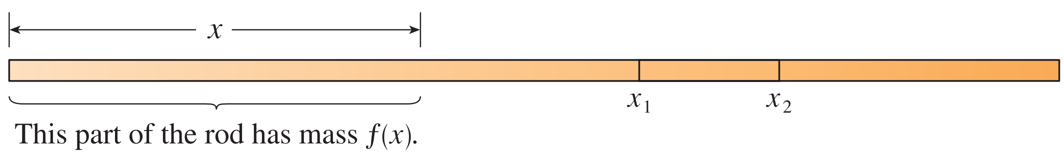
(f) Find the total distance traveled by the particle during the first five seconds.

(g) Find the acceleration at time t and after 4 s.

(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$.

(i) When is the particle speeding up? When is it slowing down?

Example 2. If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length ($\rho = m/l$) and measured in kilograms per meter. Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point x is $m = f(x)$, as shown in the figure.



In this case the average density is the average rate of change over a given interval, and the linear density is the limit of these average densities.
If $m = f(x) = \sqrt{x}$, where x is measured in meters and m in kilograms, find the average density of the part of the rod given by $1 \le x \le 1.2$ and the density at $x = 1$.

Example 3. The average current during a time interval is the average rate of change of the net charge over that interval, and the current at a given time is the limit of the average current (the rate at which charge flows through a surface, measured in units of charge per unit time). The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds) is given by $Q(t) = t^3 - 2t^2 + 6t + 2$. [The unit of current is an ampere (1 A = 1 C/s).] Find the current when
(a) $t = 0.5$ s

(b) $t = 1$ s.

At what time is the current lowest?

Example 4. The concentration of a reactant A is the number of moles (1 mole = 6.022×10^{23} molecules) per liter and is denoted by [A] for a chemical reaction

$$\mathrm{A} + \mathrm{B} \to \mathrm{C}.$$

The average rate of reaction during a time interval is the average rate of change of the concentration of the product [C] over that interval, and the rate of reaction at a given time is the limit of the average rate of reaction.
If one molecule of a product C is formed from one molecule of a reactant A and one molecule of a reactant B, and the initial concentrations of A and B have a common value $[\mathrm{A}] = [\mathrm{B}] = a$ moles/L, then

$$[\mathrm{C}] = \frac{a^2kt}{akt+1}$$

where k is a constant.

(a) Find the rate of reaction at time t.

(b) Show that if $x = [\mathrm{C}]$, then

$$\frac{dx}{dt} = k(a-x)^2.$$

(c) What happens to the concentration as $t \to \infty$?

(d) What happens to the rate of reaction as $t \to \infty$?

(e) What do the results of parts (c) and (d) mean in practical terms?

Example 5. If a given substance is kept a constant temperature, then the rate of change of its volume V with respect to its pressure P is the derivative dV/dP. The compressibility is defined by

$$\text{isothermal compressibility} = \beta = -\frac{1}{V}\frac{dV}{dP}.$$

The volume V (in cubic meters) of a sample of air at 25°C was found to be related to the pressure P (in kilopascals) by the equation

$$V = \frac{5.3}{P}.$$

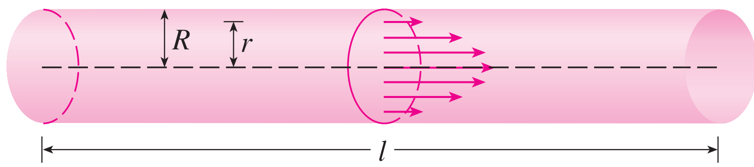
Determine the compressibility when $P = 50$ kPa.

Example 6. Let $n = f(t)$ be the number of individuals in an animal or plant population at time t. The average rate of growth during a time period is the average rate of change of the growth of the population over that time period, and the rate of growth at a given time is the limit of the average rate of growth.
Suppose that a population of bacteria doubles every hour. The population function representing the bacteria's growth can be found to be

$$n = n_0 2^t$$

where n_0 is the initial population and the time t is measured in hours.
Find the rate of growth for a colony of bacteria with an initial population $n_0 = 100$ after 4 hours.

Example 7. The shape of a blood vessel can be modeled by a cylindrical tube with radius R and length l as illustrated in the figure.



The relationship between the velocity v of the blood and the distance r from the axis is given by the law of laminar flow

$$v = \frac{P}{4\eta l}(R^2 - r^2)$$

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P and l are constant, then v is a function of r with domain $[0, R]$. The velocity gradient at a given time is the limit of the average rate of change of the velocity.
For one of the smaller human arteries we can take $\eta = 0.027$, $R = 0.008$ cm, $l = 2$ cm, and $P = 4000$ dynes/cm^2. Find the speed at which blood is flowing at $r = 0.002$ and find the velocity gradient at that point.

Example 8. Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C is called a cost function. The instantaneous rate of change of cost with respect to the number of items produced, called the marginal cost, is the limit of the average rate of change of the cost.
Suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10,000 + 5x + 0.01x^2.$$

Find the marginal cost at the production level of 500 items and compare it to the actual cost of producing the 501st item.

3.8 Exponential Growth and Decay

Definition 3.8.1. The equation

$$\frac{dy}{dt} = ky$$

is called the law of natural growth (if $k > 0$) or the law of natural decay (if $k < 0$). It is called a differential equation because it involves an unknown function y and its derivative dy/dt.

Theorem 3.8.1. *The only solutions of the differential equation* $dy/dt = ky$ *are the exponential functions*

$$y(t) = y(0)e^{kt}.$$

Definition 3.8.2. If $P(t)$ is the size of a population at time t, then

$$k = \frac{1}{P}\frac{dP}{dt}$$

is the growth rate divided by population, called the relative growth rate.

Example 1. Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

Definition 3.8.3. If $m(t)$ is the mass remaining from an initial mass m_0 of a substance after time t, then the relative decay rate is

$$-\frac{1}{m}\frac{dm}{dt}.$$

It follows that the mass decays exponentially according to the equation

$$m(t) = m_0 e^{kt},$$

where the rate of decay is expressed in terms of half-life, the time required for half of any given quantity to decay.

Example 2. The half-life of radium-226 is 1590 years.

(a) A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of the sample that remains after t years.

(b) Find the mass after 1000 years correct to the nearest milligram.

(c) When will the mass be reduced to 30 mg?

Example 3. Newton's Law of Cooling can be represented as a differential equation

$$\frac{dT}{dt} = k(T - T_s),$$

where T is the temperature of the object at time t and T_s is the temperature of the surroundings. The exponential function $y(t) = y(0)e^{kt}$ is a solution to this differential equation when $y(t) = T(t) - T_s$.
A bottle of soda pop at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F. After half an hour the soda pop has cooled to 61°F.

(a) What is the temperature of the soda pop after another half hour?

(b) How long does it take for the soda pop to cool to 50°F?

Example 4. In general, if an amount A_0 is invested at an interest rate r, then after t years it is worth $A_0(1+r)^t$. Usually, however, interest is compounded more frequently, say, n times a year. Then in each compounding period the interest rate is r/n and there are nt compounding periods in t years, so the value of the investment is

$$A_0\left(1+\frac{r}{n}\right)^{nt}.$$

Therefore, taking limits gives us the amount after t years as

$$A(t) = A_0e^{rt}$$

when interest is continuously compounded. Determine the value of an investment of \$1000 after 3 years of continuously compounding 6% interest. Compare this to the value of the same investment compounded annually instead.

3.9 Related Rates

Example 1. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Example 2. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Example 3. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

Example 4. Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

Example 5. A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

3.10 Linear Approximations and Differentials

Definition 3.10.1. The approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the linear approximation or tangent line approximation of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization of f at a.

Example 1. Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

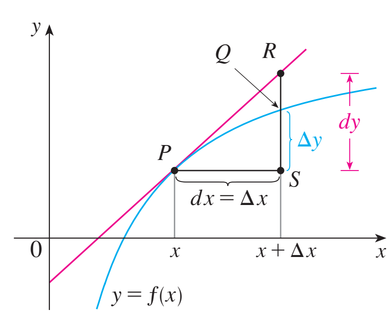
Example 2. For what values of x is the linear approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

accurate to within 0.5? What about accuracy to within 0.1?

Definition 3.10.2. If $y = f(x)$, where f is a differentiable function, then the differential dx is an independent variable; that is, dx can be given the value of any real number. The differential dy is then defined in terms of dx by the equation

$$dy = f'(x)dx.$$



Example 3. Compare the values Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes

(a) from 2 to 2.05

(b) from 2 to 2.01.

Example 4. The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

3.11 Hyperbolic Functions

Definition 3.11.1. Functions that have the same relationship to the hyperbola that trigonometric functions have to the circle are called hyperbolic functions and are defined as follows

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}.$$

Theorem 3.11.1 (Hyperbolic Identities)**.**

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$
$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$$

Example 1. Prove

(a) $\cosh^2 x - \sinh^2 x = 1$

(b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

Theorem 3.11.2 (Derivatives of Hyperbolic Functions)**.**

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$
$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x.$$

Example 2. Find $\frac{d}{dx}(\cosh\sqrt{x})$.

Theorem 3.11.3 (Inverse Hyperbolic Functions)**.**

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}) \qquad x \in \mathbb{R}$$
$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1}) \qquad x \geq 1$$
$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -1 < x < 1.$$

Example 3. Show that $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$.

Theorem 3.11.4 (Derivatives of Inverse Hyperbolic Functions)**.**

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}}$$
$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}.$$

Example 4. Prove that $\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$.

Example 5. Find $\dfrac{d}{dx}[\tanh^{-1}(\sin x)]$.

Chapter 4

Applications of Differentiation

4.1 Maximum and Minimum Values

Definition 4.1.1. Let c be a number in the domain D of a function f. Then $f(c)$ is the absolute maximum value (or global maximum value) of f on D if $f(c) \geq f(x)$ for all x in D and $f(c)$ is the absolute minimum value (or global minimum value) of f on D if $f(c) \leq f(x)$ for all x in D. These values are called extreme values of f.

Definition 4.1.2. The number $f(c)$ is a local maximum value of f if $f(c) \geq f(x)$ when x is near c and a local minimum value of f if $f(c) \leq f(x)$ when x is near c. When we say near, we mean on an open interval containing c. These values are called local extreme values of f.

Example 1. For what values of x does $f(x) = \cos x$ take on its maximum and minimum values?

Example 2. Find all of the extreme values of $f(x) = x^2$.

Example 3. Find all of the extreme values of $f(x) = x^3$.

Example 4. Find all of the extreme values of $f(x) = 3x^4 - 16x^3 + 18x^2$ within the domain $-1 \leq x \leq 4$.

Theorem 4.1.1 (Extreme Value Theorem)**.** *If f is continuous on a closed interval $[a, b]$ then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.*

Theorem 4.1.2 (Fermat's Theorem)**.** *If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.*

Proof. Suppose f has a local maximum at c. Then, by definition, $f(c) \geq f(x)$ if x is near c, so if we let $h > 0$ be close to 0 we have

$$\begin{aligned} f(c) &\geq f(c+h) \\ f(c+h) - f(c) &\leq 0 \\ \frac{f(c+h) - f(c)}{h} &\leq \frac{0}{h} \\ \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} &\leq \lim_{h \to 0^+} 0 \\ f'(c) &\leq 0. \end{aligned}$$

If $h < 0$, the direction of the inequality is reversed and we get $f'(c) \geq 0$. Thus combining these inequalities gives us $f'(c) = 0$. A similar argument can be used to achieve the same result if f has a local minimum at c. □

Example 5. Use the function $f(x) = x^3$ to determine whether the converse of Fermat's theorem is true.

Example 6. Does Fermat's theorem apply to the function $f(x) = |x|$?

Definition 4.1.3. A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 7. Find the critical numbers of $x^{3/5}(4 - x)$.

Example 8. Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \leq x \leq 4.$$

Example 9. (a) Use a graphing device to estimate the absolute minimum and maximum values of the function $f(x) = x - 2\sin x$, $0 \leq x \leq 2\pi$.

(b) Use calculus to find the exact minimum and maximum values.

Example 10. The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery*. A model for the velocity of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ seconds, is given by

$$v(t) = 0.001302t^3 - 0.09029t^2 + 23.61t - 3.083$$

(in feet per second). Using this model, estimate the absolute maximum and minimum values of the acceleration of the shuttle between liftoff and the jettisoning of the boosters.

4.2 The Mean Value Theorem

Theorem 4.2.1 (Rolle's Theorem)**.** *Let f be a function that satisfies the following three hypotheses:*

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*
3. *$f(a) = f(b)$.*

Then there is a number c in (a, b) such that $f'(c) = 0$.

Proof. If $f(x) = k$, a constant, then $f'(x) = 0$ for all $x \in (a, b)$. If $f(x) > f(a)$ for some $x \in (a, b)$ then f has a local maximum for a number $c \in (a, b)$ by the extreme value theorem. Since f is differentiable on (a, b), $f'(c) = 0$ by Fermat's theorem. By the same reasoning, $f'(c) = 0$ if $f(x) < f(a)$. □

Example 1. How could Rolle's theorem be applied to a position function that models a ball thrown upward?

Example 2. Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Theorem 4.2.2 (The Mean Value Theorem)**.** *Let f be a function that satisfies the following hypotheses:*

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Let h be the difference between f and the secant line to f on $[a, b]$, i.e.,

$$h(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right].$$

Then h is continuous on $[a, b]$ and differentiable on (a, b) because it is the sum of f and a first-degree polynomial, which are both continuous on $[a, b]$ and differentiable on (a, b). Also,

$$\begin{aligned} h(a) &= f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0 \\ h(b) &= f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = 0, \end{aligned}$$

so $h(a) = h(b)$. Therefore, by Rolle's thereom, there is a number c in (a, b) such that $h'(c) = 0$, i.e.,

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a},$$

which is equivalent to

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

as desired. □

Example 3. Find a number c in $(0, 2)$ such that the slope of the secant line is equal to the slope of the tangent line for the function $f(x) = x^3 - x$.

Example 4. What does the mean value theorem say about the velocity of an object moving in a straight line?

Example 5. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

Theorem 4.2.3. *If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).*

Proof. Let $x_1, x_2 \in (a, b)$ be such that $x_1 < x_2$. By the mean value theorem for f on $[x_1, x_2]$, we get

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1),$$

for some $c \in (x_1, x_2)$. But $f'(x) = 0$ for all x in this interval, so $f(x_2) = f(x_1)$. Since x_1 and x_2 were chosen arbitrarily, f is constant on (a, b). □

Corollary 4.2.1. *If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is $f(x) = g(x) + c$ where c is a constant.*

Proof. Let

$$F(x) = f(x) - g(x).$$

Then

$$F'(x) = f'(x) - g'(x) = 0,$$

so F is constant by the previous theorem, and thus $f - g$ is constant. □

Example 6. Prove the identity $\tan^{-1} x + \cot^{-1} x = \pi/2$.

4.3 Derivatives and the Shape of a Graph

Theorem 4.3.1 (Increasing/Decreasing Test)**.**

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Proof. Let x_1, x_2 be two numbers on an interval where $f'(x) > 0$ such that $x_1 < x_2$. Then by the mean value theorem,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c in the interval. But $f'(c) > 0$ and $x_2 - x_1 > 0$, so $f(x_2) - f(x_1) > 0$, i.e.,

$$f(x_2) > f(x_1)$$

in the interval. Since x_1 and x_2 were chosen arbitrarily, we are done, and the second half of the theorem is proved similarly. □

Example 1. Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Theorem 4.3.2 (The First Derivative Test)**.** *Suppose that c is a critical number of a continuous function f.*

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and to the right of c, or negative to the left and to the right of c, then f has no local minimum or maximum at c.

Example 2. Find the local minimum and maximum values of the function f in Example 1.

Example 3. Find the local maximum and minimum values of the function

$$g(x) = x + 2\sin x \qquad 0 \le x \le 2\pi.$$

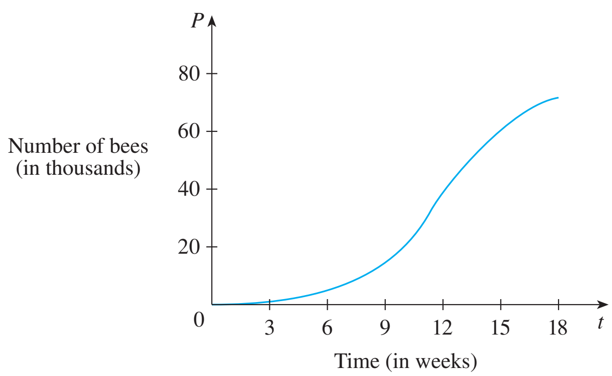
Definition 4.3.1. If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I. If the graph of f lies below all of its tangents on I, it is called concave downward on I.

Theorem 4.3.3 (Concavity Test)**.**

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Example 4. The figure shows a population graph for Cyprian honeybees raised in an apiary. How does the rate of population increase change over time? When is this rate highest? Over what intervals is P concave upward or concave downward?



Definition 4.3.2. A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Example 5. Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$.

(ii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$.

(iii) $\lim_{x\to-\infty} f(x) = -2$, $\lim_{x\to\infty} f(x) = 0$.

Theorem 4.3.4 (The Second Derivative Test)**.** *Suppose f' is continuous near c.*

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example 6. Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Example 7. Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$.

Example 8. Use the first and second derivatives of $f(x) = e^{1/x}$, together with asymptotes, to sketch its graph.

4.4 Indeterminate Forms and l'Hospital's Rule

Theorem 4.4.1 (L'Hospital's Rule). *Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that*

$$\lim_{x\to a} f(x) = 0 \qquad \textit{and} \qquad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \qquad \textit{and} \qquad \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Example 1. Find $\lim_{x\to 1} \frac{\ln x}{x-1}$.

Example 2. Calculate $\lim_{x\to\infty} \frac{e^x}{x^2}$.

Example 3. Calculate $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$.

Example 4. Find $\lim_{x\to 0} \frac{\tan x - x}{x^3}$.

Example 5. Find $\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}$.

Example 6. Evaluate $\lim_{x \to 0^+} x \ln x$.

Example 7. Compute $\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$.

Example 8. Calculate $\lim_{x \to \infty} (e^x - x)$.

Example 9. Calculate $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$.

Example 10. Find $\lim_{x \to 0^+} x^x$.

4.5 Summary of Curve Sketching

Use the following guidelines when sketching curves by hand:

A. Domain

B. Intercepts

C. Symmetry

D. Asymptotes

E. Intervals of Increase or Decrease

F. Local Maximum and Minimum Values

G. Concavity and Points of Inflection

Example 1. Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

Example 2. Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

Example 3. Sketch the graph of $f(x) = xe^x$.

Example 4. Sketch the graph of $f(x) = \dfrac{\cos x}{2 + \sin x}$.

Example 5. Sketch the graph of $y = \ln(4 - x^2)$.

Definition 4.5.1. If

$$\lim_{x\to\infty}[f(x) - (mx + b)] = 0$$

where $m \neq 0$, then the line $y = mx + b$ is called a slant asymptote because the vertical distance between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0.

Example 6. Sketch the graph of $f(x) = \dfrac{x^3}{x^2 + 1}$.

4.6 Graphing with Calculus and Calculators

Example 1. Graph the polynomial $f(x) = 2x^6 + 3x^5 + 3x^3 - 2x^2$. Use the graphs of f' and f'' to estimate all maximum and minimum points and intervals of concavity.

Example 2. Draw the graph of the function

$$f(x) = \frac{x^2 + 7x + 3}{x^2}$$

in a viewing rectangle that contains all the important features of the function. Estimate the maximum and minimum values and the intervals of concavity. Then use calculus to find these quantities exactly.

Example 3. Graph the function $f(x) = \dfrac{x^2(x+1)^3}{(x-2)^2(x-4)^4}$.

Example 4. Graph the function $f(x) = \sin(x + \sin 2x)$. For $0 \leq x \leq \pi$, estimate all maximum and minimum values, intervals of increase and decrease, and inflection points.

Example 5. How does the graph of $f(x) = 1/(x^2 + 2x + c)$ vary as c varies?

4.7 Optimization Problems

Example 1. A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Example 2. A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

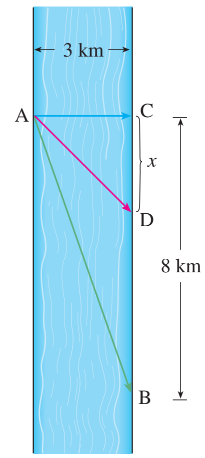
Theorem 4.7.1 (First Derivative Test for Absolute Extreme Values)**.** *Suppose that c is a critical number of a continuous function f defined on an interval.*

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Example 3. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

Example 4. A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see the figure). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)



Example 5. Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Definition 4.7.1. If $p(x)$ is the price per unit that a company can charge if it sells x units, then p is called the demand function (or price function).
If x units are sold, then the total revenue

$$R(x) = \text{quantity} \times \text{price} = xp(x)$$

and R is called the revenue function. The derivative R' of the revenue function is called the marginal revenue function and is the rate of change of revenue with respect to the number of units sold.
If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

where C is the cost function and P is called the profit function. The marginal profit function is P', the derivative of the profit function.

Example 6. A store has been selling 200 flat-screen TVs a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of TVs sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

4.8 Newton's Method

Theorem 4.8.1 (Newton's Method)**.** *If x_n is the nth approximation of a root r for a function f then*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example 1. Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

Example 2. Use Newton's method to find $\sqrt[6]{2}$ to eight decimal places.

Example 3. Find, correct to six decimal places, the root of the equation $\cos x = x$.

4.9 Antiderivatives

Definition 4.9.1. A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem 4.9.1. *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is*

$$F(x) + C$$

where C is an arbitrary constant.

Proof. Follows by Corollary 4.2.1 to the mean value theorem. □

Example 1. Find the most general antiderivative of each of the following functions.

(a) $f(x) = \sin x$

(b) $f(x) = 1/x$

(c) $f(x) = x^n$, $n \neq -1$

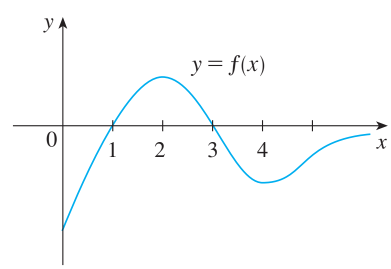
Example 2. Find all functions g such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}.$$

Example 3. Find f if $f'(x) = e^x + 20(1 + x^2)^{-1}$ and $f(0) = -2$.

Example 4. Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

Example 5. The graph of a function f is given in the figure. Make a rough sketch of an antiderivative F, given that $F(0) = 2$.



Example 6. A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

Example 7. A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground? [For motion close to the ground we may assume that the downward acceleration g is constant, its value being about 9.8 m/s^2 (or 32 ft/s^2).]

Chapter 5

Integrals

5.1 Areas and Distances

Example 1. Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

Example 2. For the region in Example 1, show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is,

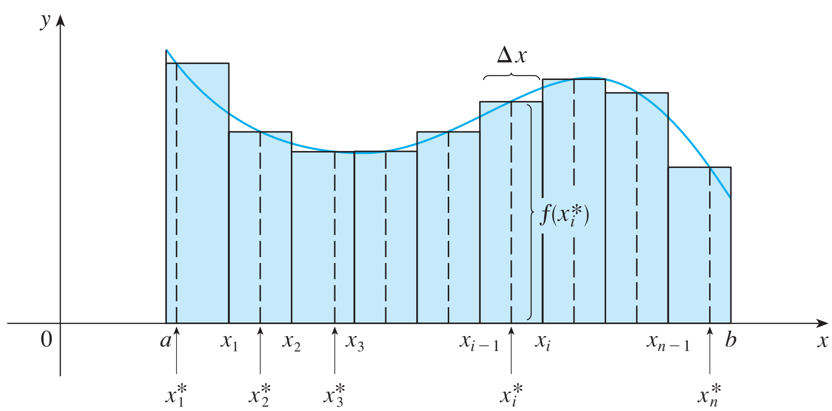
$$\lim_{n\to\infty} R_n = \frac{1}{3}.$$

Definition 5.1.1. The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

The last equality is an example of the use of sigma notation to write sums with many terms more compactly.

Definition 5.1.2. Numbers x_i^* in the ith subinterval $[x_{i-1}, x_i]$ are called sample points. We form lower (and upper) sums by choosing the sample points x_i^* so that $f(x_i^*)$ is the minimum (and maximum) value of f on the ith subinterval.



Example 3. Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

(b) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

Example 4. Suppose the odometer on a car is broken. Estimate the distance driven in feet over a 30-second time interval by using the speedometer readings taken every five seconds and recorded in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

5.2 The Definite Integral

Definition 5.2.1. If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0(= a), x_1, x_2, \ldots, x_n(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

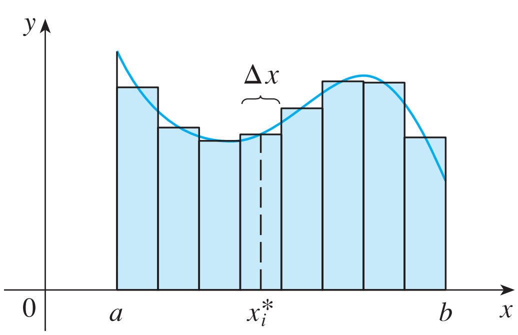
provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is integrable on $[a, b]$.

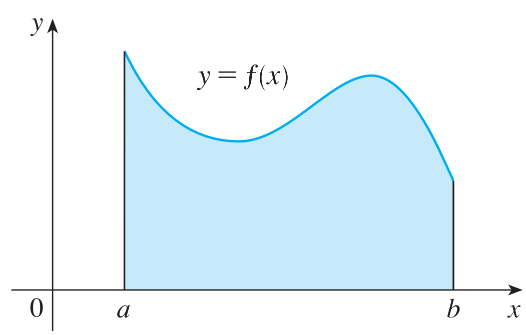
Definition 5.2.2. The symbol $\int$ is called an integral sign. In the notation $\int_a^b f(x)dx$, $f(x)$ is called the integrand and a and b are called the limits of integration; a is the lower limit and b is the upper limit. The procedure of calculating an integral is called integration.

Definition 5.2.3. The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

is called a Riemann sum and it can be used to approximate the definite integral of an integrable function within any desired degree of accuracy.

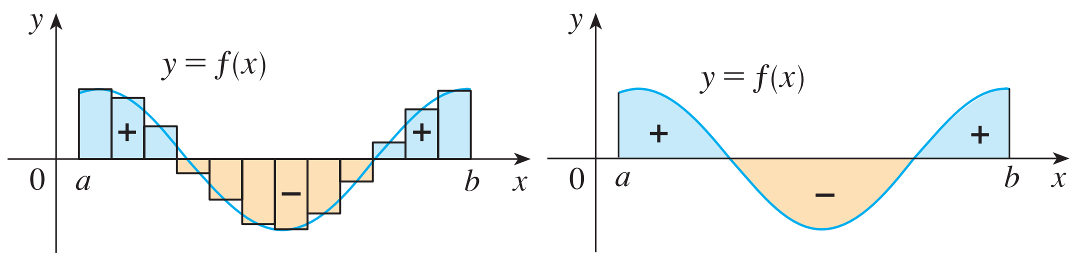




Definition 5.2.4. A definite integral can be interpreted as a net area, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and the above the graph of f.



Theorem 5.2.1. *If f is continuous on $[a,b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a,b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.*

Theorem 5.2.2. *If f is integrable on $[a,b]$, then*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_i = a + i\Delta x.$$

Example 1. Express

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x$$

as an integral on the interval $[0, \pi]$.

Theorem 5.2.3. *The following formulas are true when working with sigma notation:*

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i.$$

Example 2. (a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

(b) Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

Example 3. (a) Set up an expression for $\int_1^3 e^x \, dx$ as a limit of sums.

(b) Use a computer algebra system to evaluate the expression.

Example 4. Evaluate the following integrals by interpreting each in terms of areas.

(a) $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$

(b) $\displaystyle\int_0^3 (x-1)\,dx$

Theorem 5.2.4 (Midpoint Rule)**.**

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x[f(\bar{x}_1) + \cdots + f(\bar{x}_n)]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \textit{midpoint of } [x_{i-1}, x_i].$$

Example 5. Use the Midpoint Rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x}\,dx$.

Theorem 5.2.5 (Properties of the Definite Integral)**.**

1. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

2. $\displaystyle\int_a^a f(x)\,dx = 0.$

3. $\displaystyle\int_a^b c\,dx = c(b-a),$ *where c is any constant.*

4. $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

5. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx,$ *where c is any constant.*

6. $\displaystyle\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$

7. $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$

Example 6. Use the properties of integrals to evaluate $\displaystyle\int_0^1 (4+3x^2)\,dx.$

Example 7. If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^{8} f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$.

Theorem 5.2.6 (Comparison Properties of the Integral)**.**

8. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.

9. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

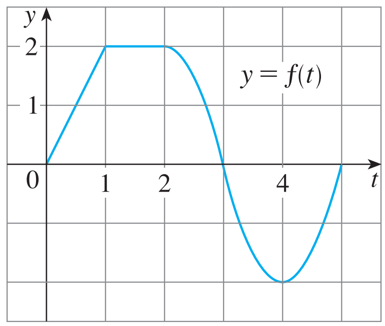
10. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Example 8. Use Property 10 to estimate $\int_0^1 e^{-x^2}\,dx$.

5.3 The Fundamental Theorem of Calculus

Example 1. If f is the function whose graph is shown in the figure and $g(x) = \int_0^x f(t)\,dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$. Then sketch a rough graph of g.



Theorem 5.3.1 (The Fundamental Theorem of Calculus, Part 1)**.** *If f is continuous on $[a, b]$, then the function g defined by*

$$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

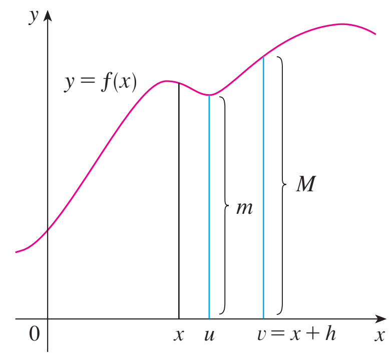
is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Proof. If x and $x + h$ are in (a, b), then

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \left(\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt \right) - \int_a^x f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt \end{aligned}$$

and so, for $h \neq 0$,

$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$



For now let's assume that $h > 0$. Since f is continuous on $[x, x+h]$, the Extreme Value Theorem says that there are numbers u and v in $[x, x+h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x, x+h]$. (See the figure.)
Then

$$\begin{aligned} mh &\leq \int_x^{x+h} f(t)\,dt \leq Mh \\ f(u)h &\leq \int_x^{x+h} f(t)\,dt \leq f(v)h \\ f(u) &\leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq f(v) \\ f(u) &\leq \frac{g(x+h)-g(x)}{h} \leq f(v). \end{aligned}$$

This inequality can be proved in a similar manner for the case where $h < 0$. Now we let $h \to 0$. Then $u \to x$ and $v \to x$, since u and v lie between x and $x+h$. Therefore

$$\lim_{h\to 0} f(u) = \lim_{u\to x} f(u) = f(x) \qquad \text{and} \qquad \lim_{h\to 0} f(v) = \lim_{u\to x} f(v) = f(x)$$

because f is continuous at x. We conclude, from the Squeeze Theorem, that

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f(x).$$

If $x = a$ or b, then this equation can be interpreted as a one-sided limit, and thus g is continuous on $[a, b]$. □

Example 2. Find the derivative of the function $g(x) = \displaystyle\int_0^x \sqrt{1+t^2}\,dt$.

Example 3. Find the derivative of the Fresnel function

$$S(x) = \int_0^x \sin(\pi t^2/2)\, dt$$

and compare its graph with that of $S(x)$ to visually confirm the fundamental theorem of calculus.

Example 4. Find $\dfrac{d}{dx}\displaystyle\int_1^{x^4} \sec t\, dt$.

Theorem 5.3.2 (The Fundamental Theorem of Calculus, Part 2)**.** *If f is continuous on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Proof. Let $g(x) = \int_a^x f(t)\,dt$. By Part 1, $g'(x) = f(x)$; that is, g is an antiderivative of f. If F is any other antiderivative of f on $[a, b]$, then, by Corollary 4.2.1,

$$F(x) = g(x) + C$$

for $a < x < b$. By continuity, this is also true for $x \in [a, b]$, so again by Part 1,

$$g(a) = \int_a^a f(t)\,dt = 0$$

and thus

$$\begin{aligned} F(b) - F(a) &= [g(b) + C] - [g(a) + C] \\ &= g(b) + C - 0 - C \\ &= g(b) \\ &= \int_a^b f(t)\,dt. \end{aligned}$$

□

Example 5. Evaluate the integral $\displaystyle\int_1^3 e^x\,dx$.

Remark 1. We often use the notation

$$F(x)\Big]_a^b = F(b) - F(a).$$

So the equation of the Fundamental Theorem of Calculus Part 2 can be written as

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \qquad \text{where} \qquad F' = f.$$

Other common notations are $F(x)|_a^b$ and $[F(x)]_a^b$.

Example 6. Find the area under the parabola $y = x^2$ from 0 to 1.

Example 7. Evaluate $\displaystyle\int_3^6 \frac{dx}{x}$.

Example 8. Find the area under the cosine curve from 0 to b, where $0 \leq b \leq \pi/2$.

Example 9. What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\, dx = \frac{x^{-1}}{-1}\Bigg]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

5.4 Indefinite Integrals and the Net Change Theorem

Definition 5.4.1. An antiderivative of f is called an indefinite integral where

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x).$$

Example 1. Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\,dx.$$

Example 2. Evaluate $\displaystyle\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$.

Example 3. Evaluate $\displaystyle\int_0^3 (x^3 - 6x)\,dx$.

Example 4. Find $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx$ and interpret the result in terms of areas.

Example 5. Evaluate $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt.$

Theorem 5.4.1 (Net Change Theorem)**.** *The integral of a rate of change is the net change:*

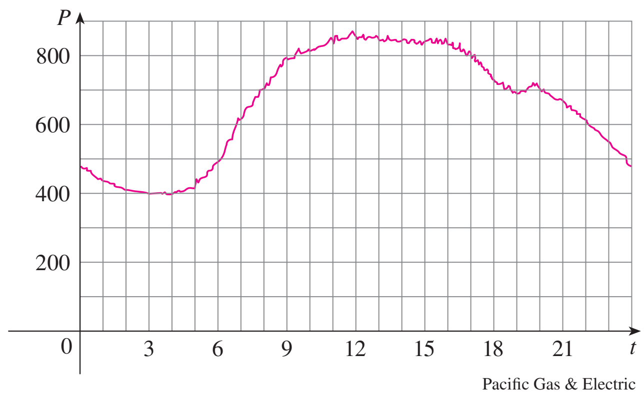
$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Example 6. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \le t \le 4$.

(b) Find the distance traveled during this time period.

Example 7. The figure shows the power consumption in the city of San Francisco for a day in September (P is measured in megawatts; t is measured in hours starting at midnight). Estimate the energy used on that day.



5.5 The Substitution Rule

Theorem 5.5.1 (The Substitution Rule)**.** *If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Proof. If $f = F'$, then, by the Chain Rule,

$$\frac{d}{dx}[F(g(x))] = f(g(x))g'(x).$$

Thus if $u = g(x)$, then we have

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C = F(u) + C = \int f(u)\,du.$$

□

Example 1. Find $\displaystyle\int x^3\cos(x^4+2)\,dx$.

Example 2. Evaluate $\displaystyle\int \sqrt{2x+1}\,dx$.

Example 3. Find $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

Example 4. Calculate $\displaystyle\int e^{5x}\,dx$.

Example 5. Find $\displaystyle\int \sqrt{1+x^2}x^5\,dx$.

Example 6. Calculate $\int \tan x \, dx$.

Theorem 5.5.2 (The Substitution Rule for Definite Integrals)**.** *If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof. Let F be an antiderivative of f. Then $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$, so by part 2 of the fundamental theorem of calculus, we have

$$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\big]_a^b = F(g(b)) - F(g(a)).$$

By applying part 2 a second time, we also have

$$\int_{g(a)}^{g(b)} f(u)\,du = F(u)\big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a)).$$ □

Example 7. Evaluate $\int_0^4 \sqrt{2x+1}\,dx$.

Example 8. Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$.

Example 9. Calculate $\int_1^e \frac{\ln x}{x}\,dx$.

Theorem 5.5.3 (Integrals of Symmetric Functions)**.** *Suppose f is continuous on $[-a, a]$.*

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\, dx = 0$.

Proof. First we split the integral:

$$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_{0}^{a} f(x)\, dx = -\int_{0}^{-a} f(x)\, dx + \int_{0}^{a} f(x)\, dx.$$

By substituting $u = -x$ we get $du = -dx$ and $u = a$ when $x = -a$, so

$$-\int_{0}^{-a} f(x)\, dx = -\int_{0}^{a} f(-u)\,(-du) = \int_{0}^{a} f(-u)\, du$$

and therefore

$$\int_{-a}^{a} f(x)\, dx = \int_{0}^{a} f(-u)\, du + \int_{0}^{a} f(x)\, dx.$$

(a) If f is even then $f(-u) = f(u)$, so

$$\int_{-a}^{a} f(x)\, dx = \int_{0}^{a} f(u)\, du + \int_{0}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx.$$

(b) If f is odd then $f(-u) = -f(u)$, so

$$\int_{-a}^{a} f(x)\, dx = -\int_{0}^{a} f(u)\, du + \int_{0}^{a} f(x)\, dx = 0.$$

□

Example 10. Evaluate $\displaystyle\int_{-2}^{2}(x^6+1)\,dx$.

Example 11. Evaluate $\displaystyle\int_{-1}^{1}\frac{\tan x}{1+x^2+x^4}\,dx$.

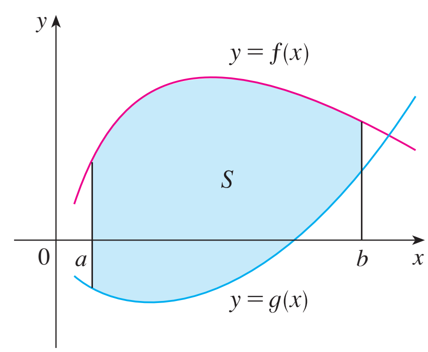
Chapter 6

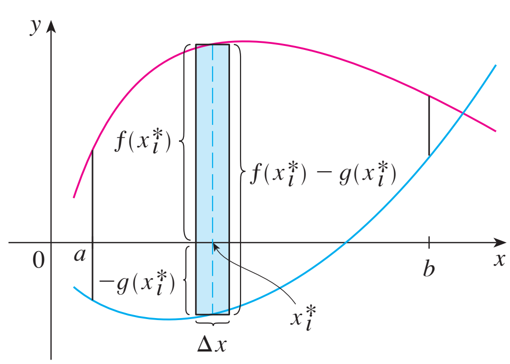
Applications of Integration

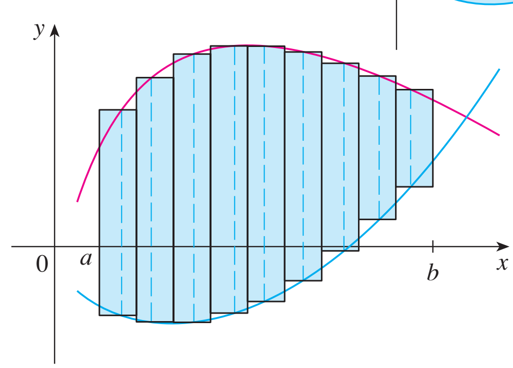
6.1 Areas Between Curves

Definition 6.1.1. The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x = \int_a^b [f(x) - g(x)]\, dx.$$





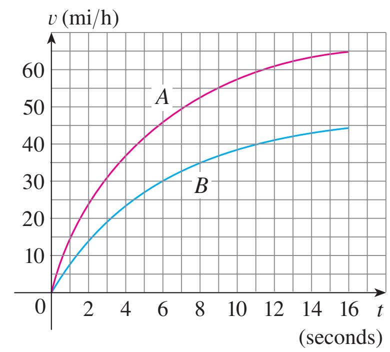


Example 1. Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

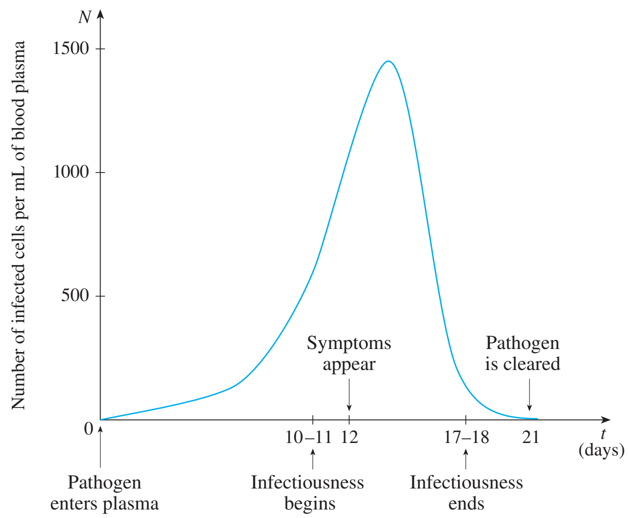
Example 2. Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Example 3. Find the approximate area of the region bounded by the curves $y = x/\sqrt{x^2+1}$ and $y = x^4 - x$.

Example 4. The figure shows the velocity curves for two cars, A and B, that start side by side and move along the same road. What does the area between the curves represent? Use the Midpoint Rule to estimate it.



Example 5. The figure is an example of a pathogenesis curve for a measles infection. It shows how the disease develops in an individual with no immunity after the measles virus spreads to the bloodstream from the respiratory tract.



The patient becomes infectious to others once the concentration of infected cells becomes great enough, and he or she remains infectious until the immune system manages to prevent further transmission. However, symptoms don't develop until the "amount of infection" reaches a particular threshold. The amount of infection needed to develop symptoms depends on both the concentration of infected cells and time, and corresponds to the area under the pathogenesis curve until symptoms appear.

(a) The pathogenesis curve in the figure has been modeled by $f(t) = -t(t-21)(t+1)$. If infectiousness begins on day $t_1 = 10$ and ends on day $t_2 = 18$, what are the corresponding concentration levels of infected cells?

(b) The level of infectiousness for an infected person is the area between $N = f(t)$ and the line through the points $P_1(t_1, (f(t_1))$ and $P_2(t_2, f(t_2))$, measured in (cells/mL)$\cdot$ days. Compute the level of infectiousness for this particular patient.

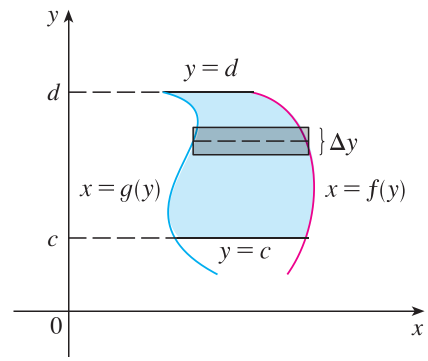
Definition 6.1.2. The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)| \, dx.$$

Example 6. Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

Remark 1. Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ (see the figure), then its area is

$$A = \int_c^d [f(y) - g(y)]\, dy.$$

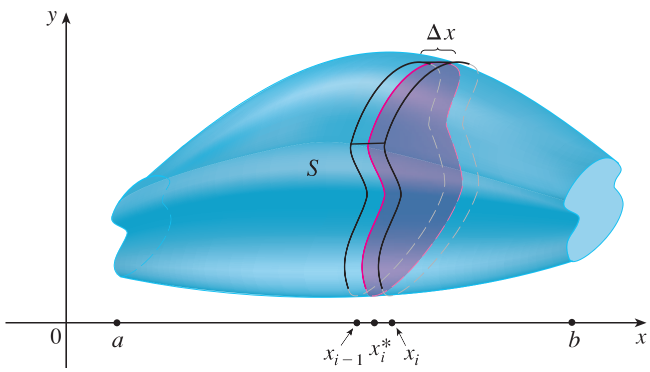


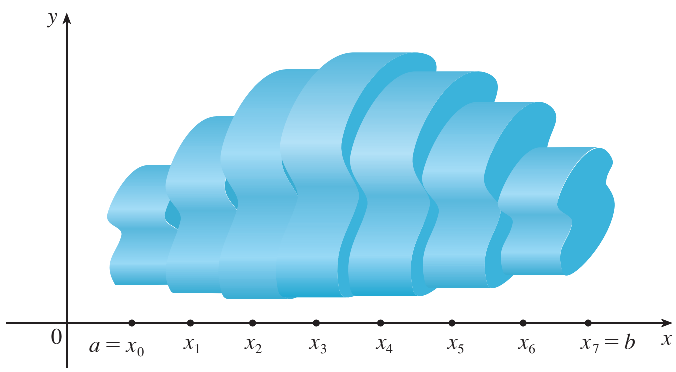
Example 7. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

6.2 Volumes

Definition 6.2.1 (Definition of Volume)**.** Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\, dx.$$





Example 1. Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Example 2. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.

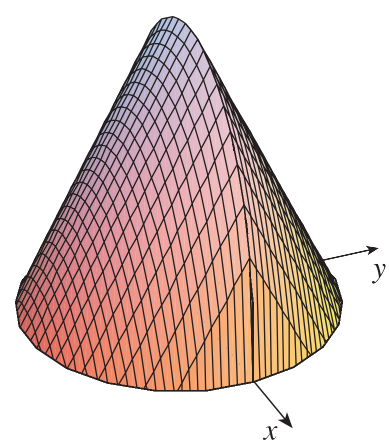
Example 3. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

Example 4. The region $\mathscr{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

Example 5. Find the volume of the solid obtained by rotating the region in Example 4 about the line $y = 2$.

Example 6. Find the volume of the solid obtained by rotating the region in Example 4 about the line $x = -1$.

Example 7. The figure shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.



Example 8. Find the volume of a pyramid whose base is a square with side L and whose height is h.

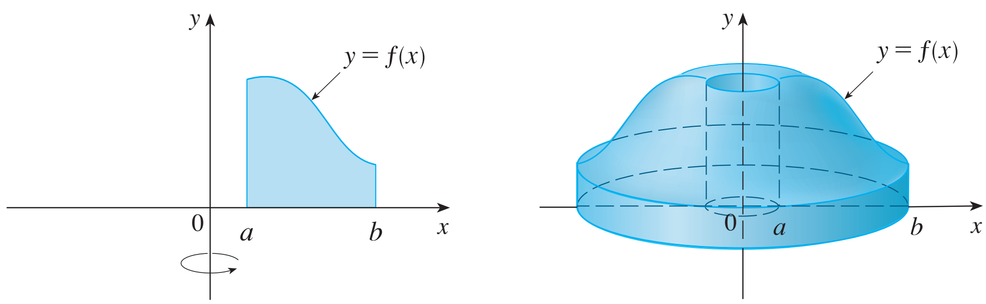
Example 9. A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.

6.3 Volumes by Cylindrical Shells

Theorem 6.3.1 (Method of Cylindrical Shells)**.** *The volume of the solid in the figure, obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is*

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\Delta x = \int_a^b 2\pi x f(x)\, dx \qquad \textit{where } 0 \leq a \leq b$$

and where $\bar{x}_i$ is the midpoint of the ith subinterval $[x_{i-1}, x_i]$.



Example 1. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Example 2. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

Example 3. Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

Example 4. Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

6.4 Work

Definition 6.4.1. In general, if an object moves along a straight line with position function $s(t)$, then the force F on the object (in the same direction) is given by Newton's Second Law of Motion as the product of its mass m and its acceleration a:

$$F = ma = m\frac{d^2s}{dt^2}.$$

Definition 6.4.2. In the case of constant acceleration, the force F is also constant and the work done is defined to be the product of the force F and distance d that the object moves:

$$W = Fd \qquad \text{work} = \text{force} \times \text{distance}.$$

Example 1. (a) How much work is done in lifting a 1.2-kg book off the floor to put it on a desk that is 0.7 m high? Use the fact that the acceleration due to gravity is $g = 9.8\ \text{m/s}^2$.

(b) How much work is done in lifting a 20-lb weight 6 ft off the ground?

Definition 6.4.3. If the force $f(x)$ on an object is variable, then we define the work done in moving the object from a to b as

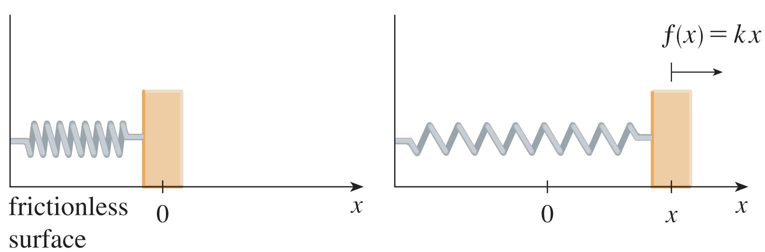
$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx.$$

Example 2. When a particle is located a distance x feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

Theorem 6.4.1 (Hooke's Law)**.** *The force required to maintain a spring stretched x units beyond its natural length is proportional to x:*

$$f(x) = kx$$

where k is a positive constant called the <u>*spring constant*</u> *(see the figure). Hooke's Law holds provided that x is not too large.*



(a) Natural position of spring (b) Stretched position of spring

Example 3. A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

Example 4. A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

Example 5. A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3.)

6.5 Average Value of a Function

Definition 6.5.1. The average value of a function f on the interval $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Example 1. Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

Theorem 6.5.1 (The Mean Value Theorem for Integrals)**.** *If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that*

$$f(c) = f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\,dx,$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Proof. By applying the Mean Value Theorem for derivatives to the function $F(x) = \int_a^x f(t)dt$, we see that there exists a number c in $[a, b]$ such that

$$\begin{aligned} F'(c) &= \frac{F(b) - F(a)}{b-a} \\ \frac{d}{dx}\left[\int_a^x f(t)\,dt\right]\Bigg|_c &= \frac{F(b) - F(a)}{b-a} \\ f(c) &= \frac{1}{b-a}[F(b) - F(a)] \\ &= \frac{1}{b-a}\int_a^b f(x)\,dx. \end{aligned}$$

□

Example 2. Find a number c in the interval $[-1, 2]$ that satisfies the mean value theorem for integrals for the function $f(x) = 1 + x^2$.

Example 3. Show that the average velocity of a car over a time interval $[t_1, t_2]$ is the same as the average of its velocities during the trip.

Chapter 7

Techniques of Integration

7.1 Integration by Parts

Theorem 7.1.1 (Formula for Integration by Parts)**.** *If f and g are differentiable functions then*

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx,$$

or, equivalently,

$$\int u\,dv = uv - \int v\,du$$

where $u = f(x)$ and $v = g(x)$.

Proof. By the Product Rule,

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= f(x)g'(x) + g(x)f'(x) \\
f(x)g(x) &= \int [f(x)g'(x) + g(x)f'(x)]\,dx \\
&= \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx \\
\int f(x)g'(x)\,dx &= f(x)g(x) - \int g(x)f'(x)\,dx
\end{aligned}$$

□

Example 1. Find $\int x \sin x \, dx$.

Example 2. Evaluate $\int \ln x \, dx$.

Example 3. Find $\int t^2 e^t \, dt$.

Example 4. Evaluate $\int e^x \sin x \, dx$.

Theorem 7.1.2 (Formula for Definite Integration by Parts)**.** *If f and g are differentiable on (a, b) and f' and g' are continuous, then*

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx.$$

Example 5. Calculate $\displaystyle\int_0^1 \tan^{-1} x\,dx$.

Example 6. Prove the reduction formula

$$\int \sin^n x\, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x\, dx$$

where $n \geq 2$ is an integer.

7.2 Trigonometric Integrals

Example 1. Evaluate $\int \cos^3 x \, dx$.

Example 2. Find $\int \sin^5 x \cos^2 x \, dx$.

Remark 1. Sometimes it is easier to use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \text{and} \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

to evaluate an integral.

Example 3. Evaluate $\int_0^\pi \sin^2 x\, dx$.

Example 4. Find $\int \sin^4 x\, dx$.

Example 5. Evaluate $\int \tan^6 x \sec^4 x \, dx$.

Example 6. Find $\int \tan^5 \theta \sec^7 \theta \, d\theta$.

Example 7. Find $\displaystyle\int \tan^3 x \, dx$.

Example 8. Find $\displaystyle\int \sec^3 x \, dx$.

Remark 2. To evaluate the integrals (a) $\int \sin mx \cos nx \, dx$, (b) $\int \sin mx \sin nx \, dx$, or (c) $\int \cos mx \cos nx \, dx$, use the corresponding identity:

$$\text{(a)}\ \sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$
$$\text{(b)}\ \sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\text{(c)}\ \cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)].$$

Example 9. Evaluate $\displaystyle\int \sin 4x \cos 5x \, dx$.

7.3 Trigonometric Substitution

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\ 0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta \le \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Example 1. Evaluate $\displaystyle\int \frac{\sqrt{9 - x^2}}{x^2}\,dx$.

Example 2. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Example 3. Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

Example 4. Find $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\,dx$.

Example 5. Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$, where $a > 0$.

Example 6. Find $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$.

Example 7. Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$.

7.4 Integration by Partial Fractions

Example 1. Find $\int \frac{x^3 + x}{x - 1}\, dx$.

Example 2. Evaluate $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx$.

Example 3. Find $\int \frac{dx}{x^2 - a^2}$, where $a \neq 0$.

Example 4. Find $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx.$

Theorem 7.4.1.

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C.$$

Example 5. Evaluate $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx.$

Example 6. Evaluate $\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\, dx$.

Example 7. Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}.$$

Example 8. Evaluate $\displaystyle\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\,dx.$

Example 9. Evaluate $\displaystyle\int \frac{\sqrt{x+4}}{x}\,dx$.

7.5 Strategy for Integration

Example 1. $\displaystyle\int \frac{\tan^3 x}{\cos^3 x}\,dx.$

Example 2. $\displaystyle\int e^{\sqrt{x}}\,dx.$

Example 3. $\displaystyle\int \frac{x^5+1}{x^3-3x^2-10x}\,dx.$

Example 4. $\displaystyle\int \frac{dx}{x\sqrt{\ln x}}$.

Example 5. $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\,dx$.

7.6 Integration Using Tables and CAS's

Example 1. The region bounded by the curves $y = \arctan x$, $y = 0$, and $x = 1$ is rotated about the y-axis. Find the volume of the resulting solid.

Example 2. Use the Table of Integrals to find $\displaystyle\int \frac{x^2}{\sqrt{5 - 4x^2}}\,dx$.

Example 3. Use the Table of Integrals to evaluate $\int x^3 \sin x \, dx$.

Example 4. Use the Table of Integrals to find $\int x\sqrt{x^2 + 2x + 4} \, dx$.

Example 5. Use a computer algebra system to find $\int x\sqrt{x^2+2x+4}\,dx$.

Example 6. Use a CAS to evaluate $\int x(x^2+5)^8\,dx$.

Example 7. Use a CAS to find $\int \sin^5 x\cos^2 x\,dx$.

7.7 Approximate Integration

Theorem 7.7.1 (Midpoint Rule)**.**

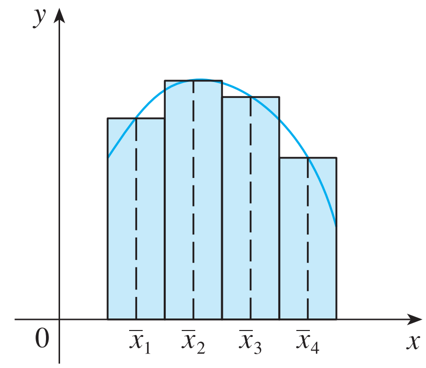
$$\int_a^b f(x)\,dx \approx M_n = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

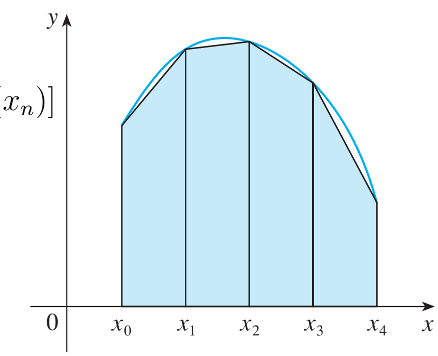
$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \textit{midpoint of } [x_{i-1}, x_i].$$



Theorem 7.7.2 (Trapezoidal Rule)**.**

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)]$$

where $\Delta x = (b-a)/n$ *and* $x_i = a + i\Delta x$.



Example 1. Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with $n = 5$ to approximate the integral $\int_1^2 (1/x)\,dx$.

Theorem 7.7.3 (Error Bounds)**.** *Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then*

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad \textit{and} \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2}.$$

Example 2. How large should we take n in order to guarantee that the Trapezoidal and Midpoint Rule approximations for $\int_1^2 (1/x)\,dx$ are accurate to within 0.0001?

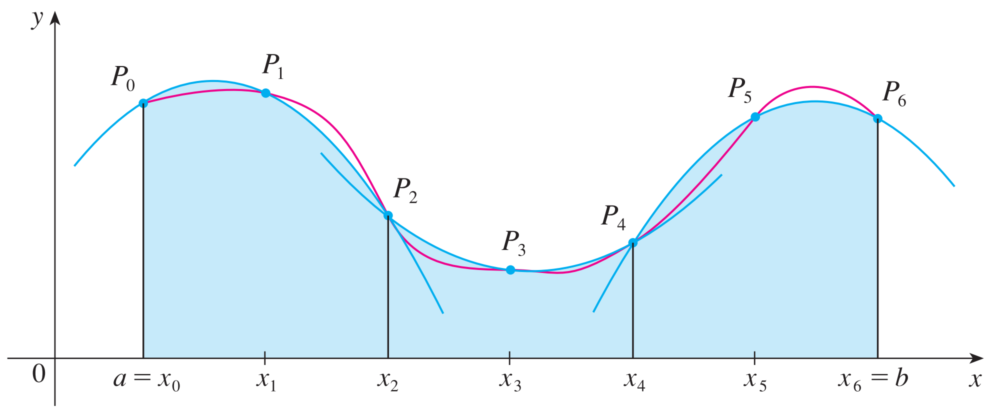
Example 3. (a) Use the Midpoint Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.

(b) Give an upper bound for the error involved in this approximation.

Theorem 7.7.4 (Simpson's Rule)**.**

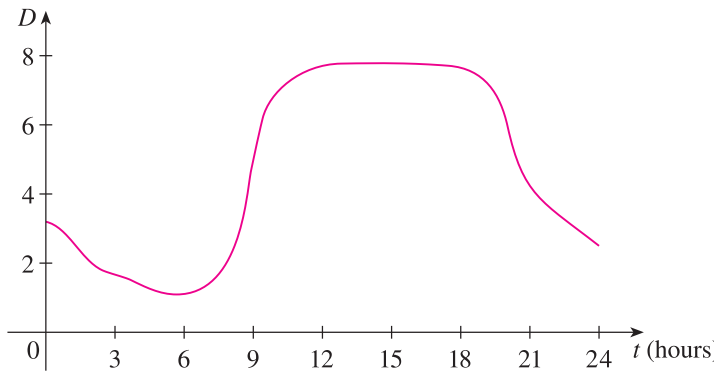
$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n is even and $\Delta x = (b-a)/n$.



Example 4. Use Simpson's Rule with $n = 10$ to approximate $\int_1^2 (1/x)\,dx$.

Example 5. The figure shows data traffic on the link from the United States to SWITCH, the Swiss academic and research network, on February 10, 1998. $D(t)$ is the data throughput, measured in megabits per second (Mb/s). Use Simpson's Rule to estimate the total amount of data transmitted on the link from midnight to noon on that day.



Theorem 7.7.5 (Error Bound for Simpson's Rule). *Suppose that* $|f^{(4)}(x)| \leq K$ *for* $a \leq x \leq b$*. If* E_S *is the error involved in using Simpson's Rule, then*

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

Example 6. How large should we take n in order to guarantee that the Simpson's Rule approximation for $\int_1^2 (1/x)\,dx$ is accurate to within 0.0001?

Example 7. (a) Use Simpson's Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.

(b) Estimate the error involved in this approximation.

7.8 Improper Integrals

Definition 7.8.1 (Definition of an Improper Integral of Type 1)**.**

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

In part (c) any real number a can be used.

Example 1. Determine whether the integral $\int_1^\infty (1/x)\,dx$ is convergent or divergent.

Example 2. Evaluate $\int_{-\infty}^{0} xe^x \, dx$.

Example 3. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$

Example 4. For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

convergent?

Definition 7.8.2 (Definition of an Improper Integral of Type 2)**.**

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called convergent if the corresponding limit exists and divergent if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example 5. Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

Example 6. Determine whether $\displaystyle\int_0^{\pi/2} \sec x\,dx$ converges or diverges.

Example 7. Evaluate $\displaystyle\int_0^3 \frac{dx}{x-1}$ if possible.

Example 8. $\int_0^1 \ln x \, dx$.

Theorem 7.8.1 (Comparison Theorem)**.** *Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.*

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Example 9. Show that $\displaystyle\int_0^\infty e^{-x^2}\,dx$ is convergent.

Example 10. Determine whether $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ converges or diverges.

Chapter 8

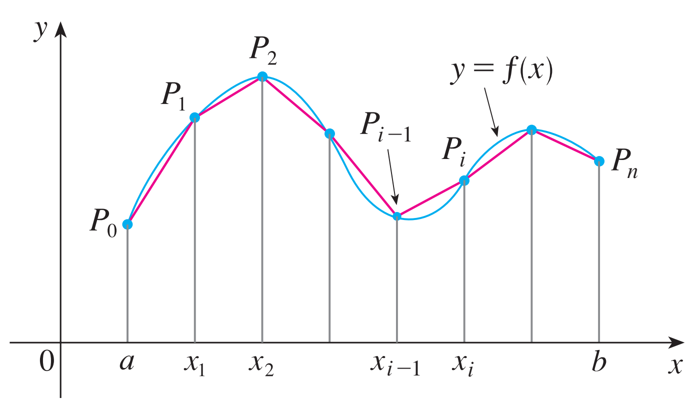
Further Applications of Integration

8.1 Arc Length

Definition 8.1.1. The <u>length</u> L of the curve C with equation $y = f(x)$, $a \leq x \leq b$, is

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

where P_i is the point $(x_i, f(x_i))$.



Theorem 8.1.1 (The Arc Length Formula)**.** *If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \le x \le b$, is*

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Proof. Let $\Delta y_i = y_i - y_{i-1}$. By the Mean Value Theorem, there is a number x_i^* between x_{i-1} and x_i such that

$$\begin{aligned} f(x_i) - f(x_{i-1}) &= f'(x_i^*)(x_i - x_{i-1}) \\ \Delta y_i &= f'(x_i^*)\Delta x. \end{aligned}$$

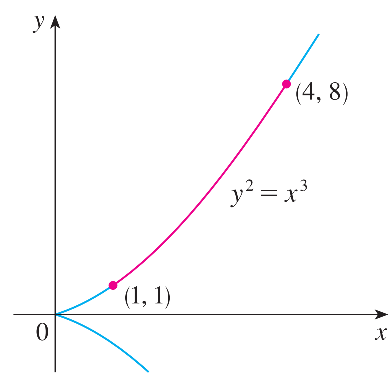
Therefore,

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2} \\ &= \sqrt{1 + [f'(x_i^*)]^2}\sqrt{(\Delta x)^2} = \sqrt{1 + [f'(x_i^*)]^2}\Delta x. \end{aligned}$$

Hence

$$\lim_{n\to\infty} \sum_{i=1}^n |P_{i-1}P_i| = \lim_{n\to\infty} \sum_{i=1}^n \sqrt{1 + [f'(x_i^*)]^2}\Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \qquad \square$$

Example 1. Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$. (See the figure.)



Remark 1. If a curve has the equation $x = g(y)$, $c \leq y \leq d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y in the Arc Length Formula, we obtain the following formula for its length:

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Example 2. Find the length of the arc of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

Example 3. (a) Set up an integral for the length of the arc of the hyperbola $xy = 1$ from the point $(1, 1)$ to the point $(2, \frac{1}{2})$.

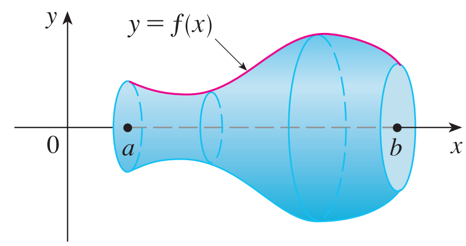
(b) Use Simpson's Rule with $n = 10$ to estimate the arc length.

Theorem 8.1.2. *If a* <u>*smooth*</u> *curve C (a curve that has a continuous derivative) has the equation $y = f(x)$, $a \le x \le b$, then $s(x)$, the distance along C from the initial point $(a, f(a))$ to the point $(x, f(x))$, is called the* <u>*arc length function*</u> *and is given by*

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

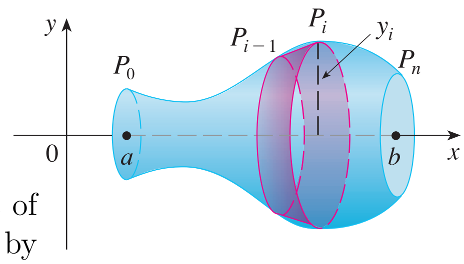
Example 4. Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $(1, 1)$ as the starting point.

8.2 Area of a Surface of Revolution

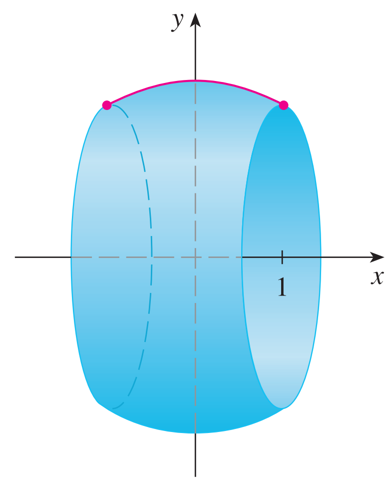


Definition 8.2.1. In the case where f is positive and has a continuous derivative, we define the surface area of the surface obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis as

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + [f'(x_i^*)]^2}\Delta x$$
$$= \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx.$$



Example 1. The curve $y = \sqrt{4 - x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis. (The surface is a portion of a sphere of radius 2. See the bottom figure.)



Example 2. The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the area of the resulting surface.

Example 3. Find the area of the surface generated by rotating the curve $y = e^x$, $0 \leq x \leq 1$, about the x-axis.

8.3 Applications to Physics and Engineering

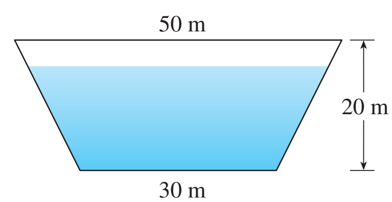
Definition 8.3.1. In general, the hydrostatic force exerted on a thin plate with area A square meters submerged in a fluid with density ρ kilograms per cubic meter at a depth d meters below the surface of the fluid is

$$F = mg = \rho g A d$$

where m is the mass and g is the acceleration due to gravity. The pressure P (in pascals) on the plate is defined to be the force per unit area:

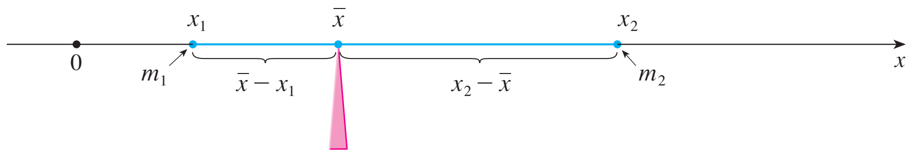
$$P = \frac{F}{A} = \rho g d.$$

Example 1. A dam has the shape of the trapezoid shown in the figure. The height is 20 m and the width is 50 m at the top and 30 m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam.



Example 2. Find the hydrostatic force on one end of a cylindrical drum with radius 3 ft if the drum is submerged in water 10 ft deep.

Definition 8.3.2. In general, for a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $x_1, x_2, \ldots, x_n$ on the x-axis,



the center of mass $\bar{x}$ is the point on which a thin plate of any given shape balances horizontally, and can be shown to be

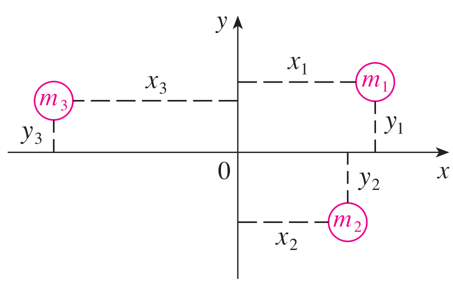
$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{m},$$

where $m_i x_i$ are called the moments of the masses m_i and $m = \sum m_i$ is the total mass of the system.
The sum of the individual moments

$$M = \sum_{i=1}^{n} m_i x_i$$

is called the moment of the system about the origin.

Definition 8.3.3. In general, for a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane



we define the moment of the system about the y-axis to be

$$M_y = \sum_{i=1}^{n} m_i x_i$$

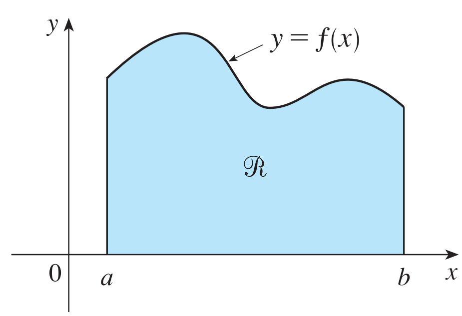
and the moment of the system about the x-axis to be

$$M_x = \sum_{i=1}^{n} m_i y_i.$$

The coordinates $(\bar{x}, \bar{y})$ of the center of mass are given by

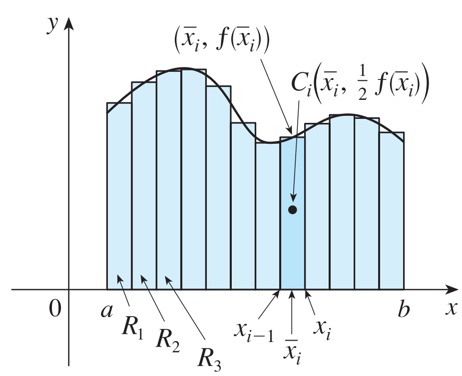
$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}.$$

Example 3. Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points $(-1, 1)$, $(2, -1)$, and $(3, 2)$, respectively.



Definition 8.3.4. The center of mass of a lamina (a flat plate) with uniform density ρ and area A that occupies a region $\mathscr{R}$ of the plane is called the centroid of $\mathscr{R}$ and is located at the point $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx.$$



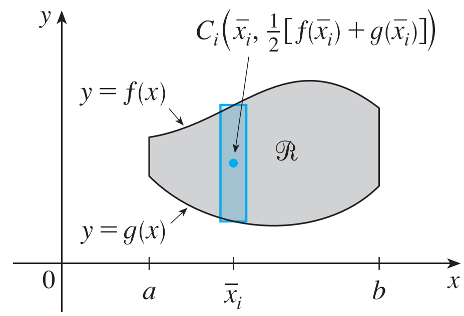
Remark 1. The symmetry principle says that if $\mathscr{R}$ is symmetric about a line l, then the centroid of $\mathscr{R}$ lies on l.

Example 4. Find the center of mass of a semicircular plate of radius r.

Example 5. Find the centroid of the region bounded by the curves $y = \cos x$, $y = 0$, $x = 0$, and $x = \pi/2$.

Theorem 8.3.1. *If the region $\mathscr{R}$ lies between two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$, then the centroid of $\mathscr{R}$ is $(\bar{x}, \bar{y})$ where*

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx$$
$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\{[f(x)]^2 - [g(x)]^2\}\,dx.$$



Example 6. Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$.

Theorem 8.3.2 (Theorem of Pappus). *Let $\mathscr{R}$ be a plane region that lies entirely on one side of a line l in the plane. If $\mathscr{R}$ is rotated about l, then the volume of the resulting solid is the product of the area A of $\mathscr{R}$ and the distance d traveled by the centroid of $\mathscr{R}$.*

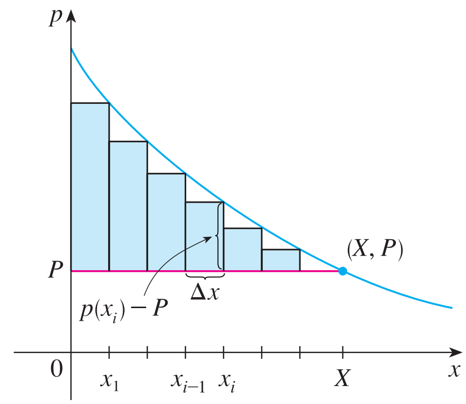
Example 7. A torus is formed by rotating a circle of radius r about a line in the plane of the circle that is a distance R $(> r)$ from the center of the circle. Find the volume of the torus.

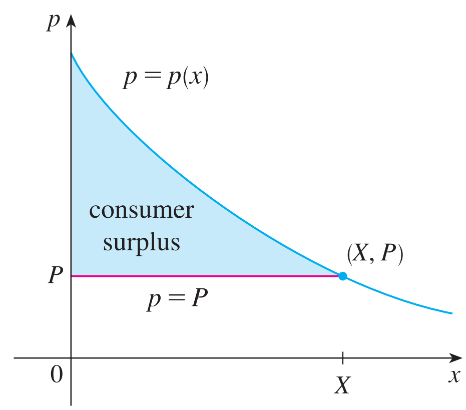
8.4 Applications to Economics and Biology

Definition 8.4.1. The consumer surplus for a commodity is defined as

$$\int_0^X [p(x) - P]\,dx$$

where $p(x)$ is the demand function, and P is the current selling price for the amount of the commodity X that can currently be sold.





Example 1. The demand for a product, in dollars, is

$$p = 1200 - 0.2x - 0.0001x^2.$$

Find the consumer surplus when the sales level is 500.

Definition 8.4.2. The cardiac output of the heart is the volume of blood pumped by the heart per unit time, that is, the rate of flow into the aorta. It is given by

$$F = \frac{A}{\int_0^T c(t)\,dt}$$

where A is the amount of dye injected into the right atrium, $[0, T]$ is the time interval until the dye has cleared, and $c(t)$ is the concentration of the dye at time t.

Example 2. A 5-mg bolus of dye is injected into a right atrium. The concentration of the dye (in milligrams per liter) is measured in the aorta at one-second intervals as shown in the table. Estimate the cardiac output.

t	$c(t)$
0	0
1	0.4
2	2.8
3	6.5
4	9.8
5	8.9
6	6.1
7	4.0
8	2.3
9	1.1
10	0

8.5 Probability

Definition 8.5.1. The probability density function f of a continuous random variable X (a quantity whose values range over an interval of real numbers) is given by:

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

where $f(x) \geq 0$ for all x and

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Example 1. Let $f(x) = 0.006x(10 - x)$ for $0 \leq x \leq 10$ and $f(x) = 0$ for all other values of x.

(a) Verify that f is a probability density function.

(b) Find $P(4 \leq X \leq 8)$

Example 2. Phenomena such as waiting times and equipment failure times are commonly modeled by exponentially decreasing probability density functions. Find the exact form of such a function.

Definition 8.5.2. In general, the mean of any probability density function f is defined to be

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx.$$

Example 3. Find the mean of the exponential distribution of Example 2:

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ ce^{-ct} & \text{if } t \geq 0. \end{cases}$$

Example 4. Suppose the average waiting time for a customer's call to be answered by a company representative is five minutes.

(a) Find the probability that a call is answered during the first minute, assuming that an exponential distribution is appropriate.

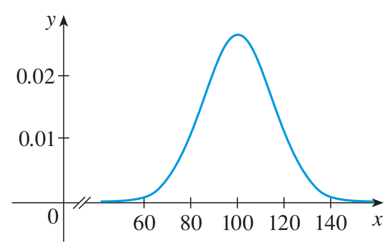
(b) Find the probability that a customer waits more than five minutes to be answered.

Definition 8.5.3. When random phenomena are modeled by a normal distribution this means that the probability density function of the random variable X is a member of the family of functions

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$$

where the positive constant σ is called the standard deviation (a measure of how spread out the values of X are).

Example 5. Intelligence Quotient (IQ) scores are distributed normally with mean 100 and standard deviation 15. (The figure shows the corresponding probability density function.)



(a) What percentage of the population has an IQ score between 85 and 115?

(b) What percentage of the population has an IQ above 140?

Chapter 9

Differential Equations

9.1 Modeling with Differential Equations

Definition 9.1.1. In general, a differential equation is an equation that contains an unknown function and one or more of its derivatives. The order of a differential equation is the order of the highest derivative that occurs in the equation. A function f is called a solution of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

Example 1. Show that every member of the family of functions

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$.

Example 2. Find a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$ that satisfies the initial condition $y(0) = 2$.

9.2 Direction Fields and Euler's Method

Definition 9.2.1. In general, suppose we have a first-order differential equation of the form

$$y' = F(x, y)$$

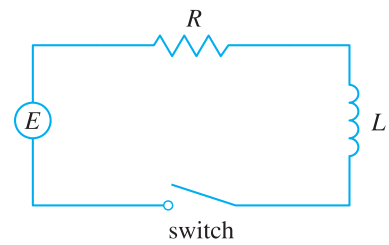
where $F(x, y)$ is some expression in x and y. If we draw short line segments with slope $F(x, y)$ at several points (x, y), the result is called a direction field (or slope field).

Example 1.

(a) Sketch the direction field for the differential equation $y' = x^2 + y^2 - 1$.

(b) Use part (a) to sketch the solution curve that passes through the origin.

Example 2. Suppose that in the simple circuit of the figure the resistance is 12 Ω, the inductance is 4 H, and a battery gives a constant voltage of 60 V.



(a) Draw a direction field for

$$L\frac{dI}{dt} + RI = E(t)$$

with these values.

(b) What can you say about the limiting value of the current?

(c) Identify any equilibrium solutions.

(d) If the switch is closed when $t = 0$ so the current starts with $I(0) = 0$, use the direction field to sketch the solution curve.

Theorem 9.2.1 (Euler's Method)**.** *Approximate values for the solution of the initial-value problem* $y' = F(x, y)$, $y(x_0) = y_0$ *with step size* h, *at* $x_n = x_{n-1} + h$, *are*

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \qquad n = 1, 2, 3, \ldots.$$

Example 3. Use Euler's method with step size 0.1 to construct a table of approximate values for the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1.$$

Example 4. In Example 2 we discussed a simple electric circuit with resistance 12 Ω, inductance 4 H, and a battery with voltage 60 V. If the switch is closed when $t = 0$, we modeled the current I at time t by the initial-value problem

$$\frac{dI}{dt} = 15 - 3I \qquad I(0) = 0.$$

Estimate the current in the circuit half a second after the switch is closed.

9.3 Separable Equations

Definition 9.3.1. A separable equation is a first-order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y. In other words, it can be written in the form

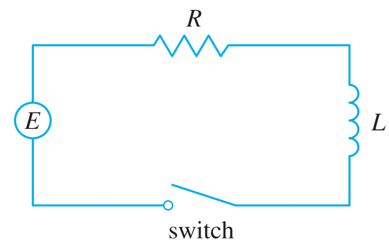
$$\frac{dy}{dx} = g(x)g(y).$$

Example 1. (a) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

Example 2. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$.

Example 3. Solve the equation $y' = x^2 y$.

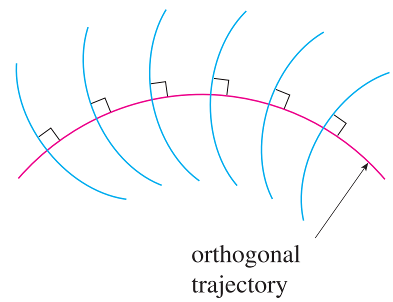


Example 4. In Section 9.2 we modeled the current $I(t)$ in the electric circuit shown in the figure by the differential equation

$$L\frac{dI}{dt} + RI = E(t).$$

Find an expression for the current in a circuit where the resistance is 12 V, the inductance is 4 H, a battery gives a constant voltage of 60 V, and the switch is turned on when $t = 0$. What is the limiting value of the current?

Definition 9.3.2. An orthogonal trajectory of a family of curves is a curve that intersects each curve of the family orthogonally, that is, at right angles (see the figure).



Example 5. Find the orthogonal trajectories of the family of curves $x = ky^2$, where k is an arbitrary constant.

Example 6. A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

9.4 Models for Population Growth

Definition 9.4.1. In general, if $P(t)$ is the value of a quantity y at time t and if the rate of change of P with respect to t is proportional to its size $P(t)$ at any time, then

$$\frac{dP}{dt} = kP$$

where k is a constant. This equation is sometimes called the law of natural growth.

Theorem 9.4.1. *The solution of the initial-value problem*

$$\frac{dP}{dt} = kP \qquad P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}.$$

Proof. The law of natural growth is a separable differential equation, so

$$\begin{aligned} \frac{dP}{dt} &= kP \\ \int \frac{dP}{P} &= \int k\,dt \\ \ln|P| &= kt + C \\ |P| &= e^{kt+C} = e^C e^{kt} \\ P &= Ae^{kt}, \end{aligned}$$

where A ($= \pm e^C$ or 0) is an arbitrary constant. Since $P(0) = A$, $P(t) = P_0 e^{kt}$. □

Definition 9.4.2. The model for population growth known as the logistic differential equation is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

where M is the carrying capacity, the maximum population that the environment is capable of sustaining in the long run.

Example 1. Draw a direction field for the logistic equation with $k = 0.08$ and carrying capacity $M = 1000$. What can you deduce about the solutions?

Theorem 9.4.2. *The solution to the logistic equation is*

$$P(t) = \frac{M}{1 + Ae^{-kt}} \qquad \textit{where } A = \frac{M - P_0}{P_0}.$$

Proof. The logistic equation is separable, so using partial fractions, we get

$$\begin{aligned}
\frac{dP}{dt} &= kP\left(1 - \frac{P}{M}\right) \\
\int \frac{dP}{P(1 - P/M)} &= \int k\,dt \\
\int \frac{M}{P(M-P)}\,dP &= \int k\,dt \\
\int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP &= \int k\,dt \\
\ln|P| - \ln|M-P| &= kt + C \\
\ln\left|\frac{M-P}{P}\right| &= -kt - C \\
\left|\frac{M-P}{P}\right| &= e^{-kt-C} = e^{-C}e^{-kt} \\
\frac{M-P}{P} &= Ae^{-kt} \\
\frac{M}{P} - 1 &= Ae^{-kt} \\
\frac{M}{P} &= 1 + Ae^{-kt} \\
P &= \frac{M}{1 + Ae^{-kt}},
\end{aligned}$$

where $A = \pm e^{-C}$. If $t = 0$, we have

$$\frac{M - P_0}{P_0} = Ae^0 = A.$$

□

Example 2. Write the solution of the initial-value problem

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) \qquad P(0) = 100$$

and use it to find the population sizes $P(40)$ and $P(80)$. At what time does the population reach 900?

Example 3. In the 1930s the biologist G. F. Gause conducted an experiment with the protozoan *Paramecium* and used a logistic equation to model his data. The table gives his daily count of the population of protozoa. He estimated the initial relative growth rate to be 0.7944 and the carrying capacity to be 64.

t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57

Find the exponential and logistic models for Gause's data. Compare the predicted values with the observed values and comment on the fit.

9.5 Linear Equations

Definition 9.5.1. A first-order linear differential equation is one that can be put into the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions on a given interval.

Theorem 9.5.1. *To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $I(x) = e^{\int P(x)dx}$ and integrate both sides.*

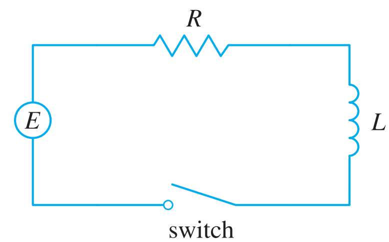
Example 1. Solve the differential equation $\dfrac{dy}{dx} + 3x^2y = 6x^2$.

Example 2. Find the solution of the initial-value problem

$$x^2y' + xy = 1 \qquad x > 0 \qquad y(1) = 2.$$

Example 3. Solve $y' + 2xy = 1$.

Example 4. Suppose that in the simple circuit of the figure the resistance is 12 V and the inductance is 4 H. If a battery gives a constant voltage of 60 V and the switch is closed when $t = 0$ so the current starts with $I(0) = 0$, find



(a) $I(t)$,

(b) the current after 1 second, and

(c) the limiting value of the current.

Example 5. Suppose that the resistance and inductance remain as in Example 4 but, instead of the battery, we use a generator that produces a variable voltage of $E(t) = 60 \sin 30t$ volts. Find $I(t)$.

9.6 Predator-Prey Systems

Definition 9.6.1. The equations

$$\frac{dR}{dt} = kR - aRW \qquad \frac{dW}{dt} = -rW + bRW$$

are known as the predator-prey equations, or the Lotka-Volterra equations. A solution of this system of equations is a pair of functions $R(t)$ and $W(t)$ that describe the populations of prey and predators as functions of time.

Example 1. Suppose that populations of rabbits and wolves are described by the Lotka-Volterra equations with $k = 0.08$, $a = 0.001$, $r = 0.02$, and $b = 0.00002$. The time t is measured in months.

(a) Find the constant solutions (called the equilibrium solutions) and interpret the answer.

(b) Use the system of differential equations to find an expression for dW/dR.

(c) Draw a direction field for the resulting differential equation in the RW-plane. Then use that direction field to sketch some solution curves.

(d) Suppose that, at some point in time, there are 1000 rabbits and 40 wolves. Draw the corresponding solution curve and use it to describe the changes in both population levels.

(e) Use part (d) to make sketches of R and W as functions of t.

Chapter 10

Parametric Equations and Polar Coordinates

10.1 Curves Defined by Parametric Equations

Definition 10.1.1. Suppose that x and y are both given as functions of a third variable t (called a parameter) by the equations

$$x = f(t) \qquad y = g(t)$$

(called parametric equations). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a parametric curve.

Example 1. Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1.$$

Definition 10.1.2. In general, the curve with parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \leq t \leq b$$

has initial point $(f(a), g(a))$ and terminal point $(f(b), g(b))$.

Example 2. What curve is represented by the following parametric equations?

$$x = \cos t \qquad y = \sin t \qquad 0 \leq t \leq 2\pi.$$

Example 3. What curve is represented by the given parametric equations?

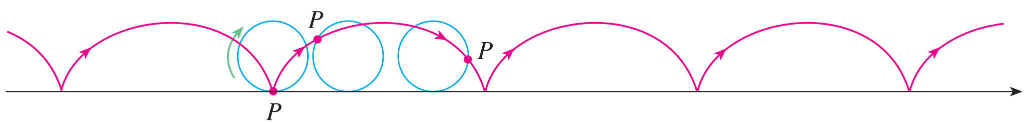
$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leq t \leq 2\pi.$$

Example 4. Find parametric equations for the circle with center (h, k) and radius r.

Example 5. Sketch the curve with parametric equations $x = \sin t$, $y = \sin^2 t$.

Example 6. Use a graphing device to graph the curve $x = y^4 - 3y^2$.

Example 7. The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a cycloid (see the figure). If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.



Example 8. Investigate the family of curves with parametric equations

$$x = a + \cos t \qquad y = a \tan t + \sin t.$$

What do these curves have in common? How does the shape change as a increases?

10.2 Calculus with Parametric Curves

Theorem 10.2.1. *Suppose f and g are differentiable functions. Then for a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x, we have*

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \textit{if } \frac{dx}{dt} \neq 0.$$

Proof. Since y is a differentiable function of x, we have, by the Chain Rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Then if $\frac{dx}{dt} \neq 0$ we can divide by it, so

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

□

Theorem 10.2.2. *Suppose f and g are differentiable functions. Then for a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x, we have*

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \qquad \textit{if } \frac{dx}{dt} \neq 0.$$

Proof. By the previous theorem,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \qquad \text{if } \frac{dx}{dt} \neq 0.$$

□

Example 1. A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

(a) Show that C has two tangents at the point $(3, 0)$ and find their equations

(b) Find the points on C where the tangent is horizontal or vertical.

(c) Determine where the curve is concave upward or downward.

(d) Sketch the curve.

Example 2.

(a) Find the tangent to the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ at the point where $\theta = \pi/3$.

(b) At what points is the tangent horizontal? When is it vertical?

Theorem 10.2.3. *If a curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$, then the area under the curve is given by*

$$A = \int_\alpha^\beta g(t) f'(t)\, dt \qquad \left[\text{or } \int_\beta^\alpha g(t) f'(t)\, dt \right].$$

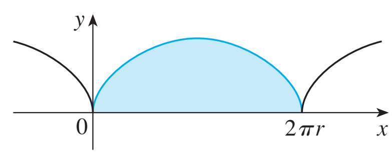
Proof. Since the area under the curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\, dx$, we can use the Substitution Rule for Definite Integrals with $y = g(t)$ and $dx = f'(t)\, dt$ to get

$$A = \int_a^b y\, dx = \int_\alpha^\beta g(t) f'(t)\, dt. \qquad \square$$

Example 3. Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta).$$

(See the figure.)



Theorem 10.2.4. *If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is*

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example 4. (a) Use the representation of the unit circle given by

$$x = \cos t \qquad y = \sin t \qquad 0 \leq t \leq 2\pi$$

to find its arc length.

(b) Use the representation of the unit circle given by

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leq t \leq 2\pi$$

to find its arc length.

Example 5. Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

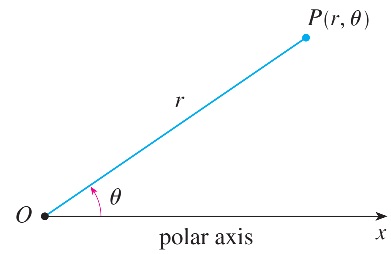
Theorem 10.2.5. *Suppose a curve C is given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f', g' are continuous, $g'(t) \geq 0$, is rotated about the x-axis. If C is traversed exactly once as t increases from α to β, then the area of the resulting surface is given by*

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example 6. Show that the surface area of a sphere of radius r is $4\pi r^2$.

10.3 Polar Coordinates

Definition 10.3.1. The polar coordinate system consists of a point called the pole (or origin) O, a ray starting at the pole called the polar axis, and other points P represented by (r, θ) where r is the distance from O to P and θ is the angle (usually measured in radians) between the polar axis and the line OP as in the figure. r, θ are called polar coordinates of P.



Example 1. Plot the points whose polar coordinates are given.

(a) $(1, 5\pi/4)$

(b) $(2, 3\pi)$

(c) $(2, -2\pi/3)$

(d) $(-3, 3\pi/4)$

Theorem 10.3.1. *If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then*

$$x = r\cos\theta \qquad y = r\sin\theta$$

and

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}.$$

Example 2. Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

Example 3. Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

Example 4. What curve is represented by the polar equation $r = 2$?

Example 5. Sketch the polar curve $\theta = 1$.

Example 6. (a) Sketch the curve with polar equation $r = 2\cos\theta$.

(b) Find a Cartesian equation for this curve.

Example 7. Sketch the curve $r = 1 + \sin\theta$.

Example 8. Sketch the curve $r = \cos 2\theta$.

Theorem 10.3.2. *The slope of the tangent line to a polar curve $r = f(\theta)$ is*

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Proof. Regard θ as a parameter and write

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Then by Theorem 10.2.1 and the product rule, we have

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}.$$ □

Example 9.

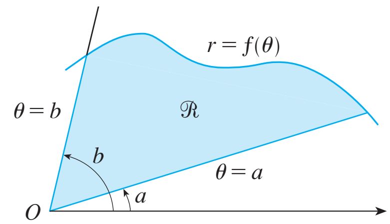
(a) For the cardioid $r = 1 + \sin\theta$ of Example 7, find the slope of the tangent line when $\theta = \pi/3$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

Example 10. Graph the curve $r = \sin(8\theta/5)$.

Example 11. Investigate the family of polar curves given by $r = 1 + c\sin\theta$. How does the shape change as c changes? (These curves are called limaçons, after a French word for snail, because of the shape of the curves for certain values of c.)

10.4 Areas and Lengths in Polar Coordinates



Theorem 10.4.1. *Let* $\mathscr{R}$ *be the region, illustrated in the figure, bounded by the polar curve* $r = f(\theta)$ *and by the rays* $\theta = a$ *and* $\theta = b$*, where* f *is a positive continuous function and where* $0 < b - a \leq 2\pi$*. The area* A *of the polar region* $\mathscr{R}$ *is*

$$A = \int_a^b \frac{1}{2} r^2 \, d\theta.$$

Example 1. Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Example 2. Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Example 3. Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

Theorem 10.4.2. *The length of a curve with polar equation $r = f(\theta)$, $a \leq \theta \leq b$, is*

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Proof. Regard θ as a parameter and write

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Then by the product rule, we have

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \qquad \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta.$$

Since $\cos^2\theta + \sin^2\theta = 1$,

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2,\end{aligned}$$

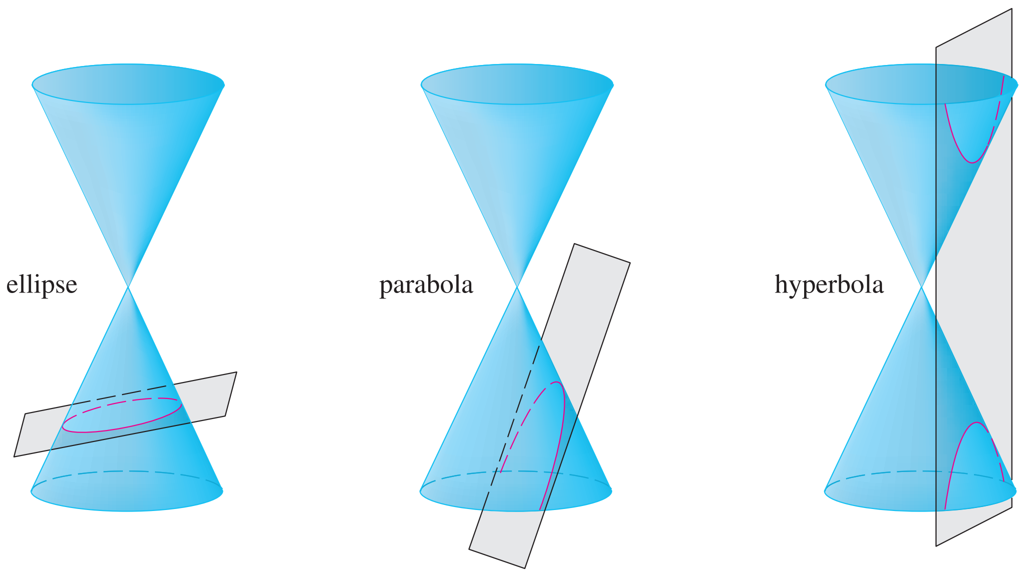
so

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \qquad \square$$

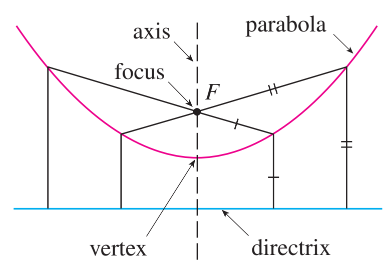
Example 4. Find the length of the cardioid $r = 1 + \sin\theta$.

10.5 Conic Sections

Definition 10.5.1. Parabolas, ellipses, and hyperbolas are called conic sections, or conics, because they result from intersecting a cone with a plane as shown in the figure.



Definition 10.5.2. A parabola is the set of points in a plane that are equidistant from a fixed point F (called the focus) and a fixed line (called the directrix). This definition is illustrated by the figure. Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the vertex. The line through the focus perpendicular to the directrix is called the axis of the parabola.



Theorem 10.5.1. *An equation of the parabola with focus* $(0, p)$ *and directrix* $y = -p$ *is*

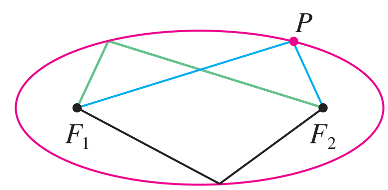
$$x^2 = 4py.$$

Theorem 10.5.2. *An equation of the parabola with focus* $(p, 0)$ *and directrix* $x = -p$ *is*

$$y^2 = 4px.$$

Example 1. Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

Definition 10.5.3. An ellipse is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant (see the figure). These two fixed points are called the foci (plural of focus).



Definition 10.5.4. If $(-c, 0)$ and $(c, 0)$ are the foci of an ellipse, the sum of the distances from a point on the ellipse to the foci are $2a > 0$, and $b^2 = a^2 - c^2$, then the points $(a, 0)$ and $(-a, 0)$ are called the vertices of ellipse and the line segment joining the vertices is called the major axis. The line segment joining $(0, b)$ and $(0, -b)$ is the minor axis.

Theorem 10.5.3. *The ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \geq b > 0$$

has foci $(\pm c, 0)$*, where* $c^2 = a^2 - b^2$*, and vertices* $(\pm a, 0)$*.*

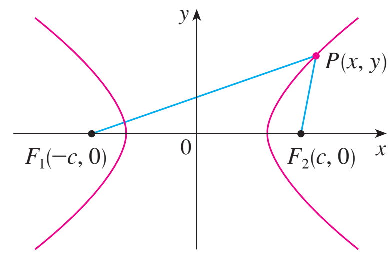
Theorem 10.5.4. *The ellipse*

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \geq b > 0$$

has foci $(0, \pm c)$*, where* $c^2 = a^2 - b^2$*, and vertices* $(0, \pm a)$*.*

Example 2. Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

Example 3. Find an equation of the ellipse with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.



Definition 10.5.5. A hyperbola is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the foci) is a constant. This definition is illustrated in the figure.

Theorem 10.5.5. *The hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci $(\pm c, 0)$*, where* $c^2 = a^2 + b^2$*, vertices* $(\pm a, 0)$*, and asymptotes* $y = \pm(b/a)x$*.*

Theorem 10.5.6. *The hyperbola*

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has foci $(0, \pm c)$*, where* $c^2 = a^2 + b^2$*, vertices* $(0, \pm a)$*, and asymptotes* $y = \pm(a/b)x$*.*

Example 4. Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

Example 5. Find the foci and equation of the hyperbola with vertices $(0, \pm 1)$ and asymptote $y = 2x$.

Example 6. Find an equation of the ellipse with foci $(2, -2)$, $(4, -2)$, and vertices $(1, -2)$, $(5, -2)$.

Example 7. Sketch the conic $9x^2 - 4y^2 - 72x + 8y + 176 = 0$ and find its foci.

10.6 Conic Sections in Polar Coordinates

Theorem 10.6.1. *Let F be a fixed point (called the focus) and l be a fixed line (called the directrix) in a plane. Let e be a fixed positive number (called the eccentricity). The set of all points P in the plane such that*

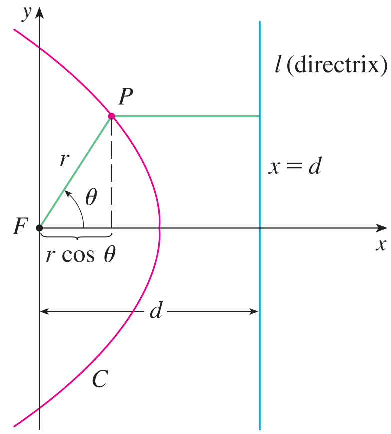
$$\frac{|PF|}{|Pl|} = e$$

(that is, the ratio of the distance from F to the distance from l is the constant e) is a conic section. The conic is

(a) an ellipse if $e < 1$

(b) a parabola if $e = 1$

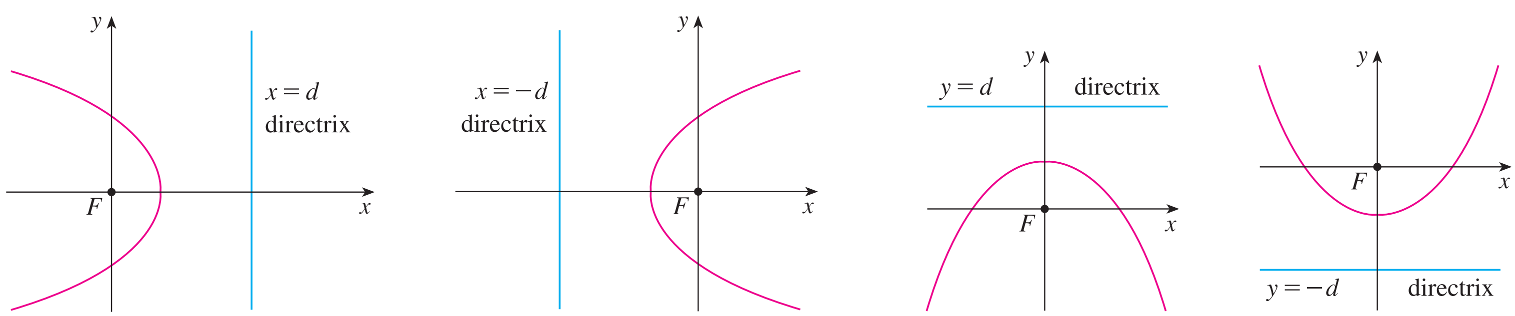
(c) a hyperbola if $e > 1$



Theorem 10.6.2. *A polar equation of the form*

$$r = \frac{ed}{1 \pm e\cos\theta} \qquad or \qquad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.



(a) $r = \dfrac{ed}{1 + e\cos\theta}$ (b) $r = \dfrac{ed}{1 - e\cos\theta}$ (c) $r = \dfrac{ed}{1 + e\sin\theta}$ (d) $r = \dfrac{ed}{1 - e\sin\theta}$

Example 1. Find a polar equation for a parabola that has its focus at the origin and whose directrix is the line $y = -6$.

Example 2. A conic is given by the polar equation

$$r = \frac{10}{3 - 2\cos\theta}.$$

Find the eccentricity, identify the conic, locate the directrix, and sketch the conic.

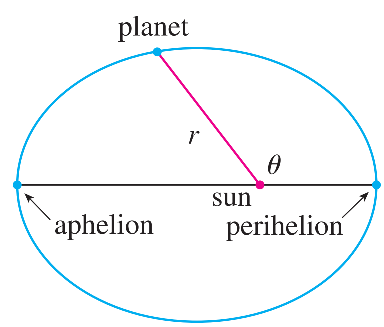
Example 3. Sketch the conic $r = \dfrac{12}{2 + 4\sin\theta}$.

Example 4. If the ellipse of Example 2 is rotated through an angle $\pi/4$ about the origin, find a polar equation and graph the resulting ellipse.

Theorem 10.6.3. *The polar equation of an ellipse with focus at the origin, semimajor axis a, eccentricity e, and directrix $x = d$ can be written in the form*

$$r = \frac{a(1-e^2)}{1+e\cos\theta}.$$

Definition 10.6.1. The positions of a planet that are closest to and farthest from the sun are called its perihelion and aphelion, respectively, and correspond to the vertices of the ellipse (see the figure). The distances from the sun to the perihelion and aphelion are called the perihelion distance and aphelion distance, respectively.



Theorem 10.6.4. *The perihelion distance from a planet to the sun is $a(1-e)$ and the aphelion distance is $a(1+e)$.*

Proof. If the sun is at the focus F, at perihelion we have $\theta = 0$, so

$$r = \frac{a(1-e^2)}{1+e\cos 0} = \frac{a(1-e)(1+e)}{1+e} = a(1-e).$$

Similarly, at aphelion $\theta = \pi$ and $r = a(1+e)$. □

Example 5. (a) Find an approximate polar equation for the elliptical orbit of the earth around the sun (at one focus) given that the eccentricity is about 0.017 and the length of the major axis is about 2.99×10^8 km.

(b) Find the distance from the earth to the sun at perihelion and at aphelion.

Chapter 11

Infinite Sequences and Series

11.1 Sequences

Definition 11.1.1. A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots.$$

The number a_1 is called the first term, a_2 is the second term, and in general a_n is the nth term.
A sequence can also be defined as a function whose domain is the set of positive integers. However, we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.
The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}.$$

Example 1. Some sequences can be defined by giving a formula for the nth term. In the following examples we give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that n doesn't have to start at 1.

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ $\quad a_n = \dfrac{n}{n+1}$ $\quad \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots, \dfrac{n}{n+1}, \ldots\right\}$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}$ $\quad a_n = \dfrac{(-1)^n(n+1)}{3^n}$ $\quad \left\{-\dfrac{2}{3}, \dfrac{3}{9}, -\dfrac{4}{27}, \dfrac{5}{81}, \ldots, \dfrac{(-1)^n(n+1)}{3^n}, \ldots\right\}$

(c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$ $\quad a_n = \sqrt{n-3}, n \geq 3$ $\quad \left\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\right\}$

(d) $\left\{\cos\dfrac{n\pi}{6}\right\}_{n=0}^{\infty}$ $\quad a_n = \cos\dfrac{n\pi}{6}, n \geq 0$ $\quad \left\{1, \dfrac{\sqrt{3}}{2}, \dfrac{1}{2}, 0, \ldots, \cos\dfrac{n\pi}{6}, \ldots\right\}$

Example 2. Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

assuming that the pattern of the first few terms continues.

Example 3. Here are some sequences that don't have a simple defining equation.

(a) The sequence $\{p_n\}$, where p_n is the population of the world as of January 1 in the year n.

(b) If we let a_n be the digit in the nth decimal place of the number e, then $\{a_n\}$ is a well-defined sequence whose first few terms are

$$\{7, 1, 8, 2, 8, 1, 8, 2, 4, 5, \ldots\}.$$

(c) The Fibonacci sequence $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1 \qquad f_2 = 1 \qquad f_n = f_{n-1} + f_{n-2} \qquad n \geq 3.$$

Each term is the sum of the two preceding terms. The first few terms are

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits.

Definition 11.1.2. A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty}$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

Definition 11.1.3 (Precise Definition of the Limit of a Sequence)**.** A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon.$$

Theorem 11.1.1. *If* $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = a_n$ *when* n *is an integer, then* $\lim_{n\to\infty} a_n = L$.

Definition 11.1.4. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad a_n > M.$$

Theorem 11.1.2 (Limit Laws for Sequences)**.** *If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then*

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$
$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$
$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0.$$

Theorem 11.1.3 (Squeeze Theorem for Sequences)**.** *If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

Theorem 11.1.4. *If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Proof. Since $\lim_{n\to\infty} |a_n| = 0$,

$$\lim_{n\to\infty} -|a_n| = 0 = -\lim_{n\to\infty} |a_n| = 0.$$

But $-|a_n| \leq a_n \leq |a_n|$ for all n, so by the squeeze theorem for sequences, $\lim_{n\to\infty} a_n = 0$. □

Example 4. Find $\lim_{n\to\infty} \dfrac{n}{n+1}$.

Example 5. Is the sequence $a_n = \dfrac{n}{\sqrt{10+n}}$ convergent or divergent?

Example 6. Calculate $\lim_{n\to\infty} \dfrac{\ln n}{n}$.

Example 7. Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Example 8. Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

Theorem 11.1.5. *If* $\lim_{n\to\infty} a_n = L$ *and the function* f *is continuous at* L*, then*

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Proof. Choose a particular n, say n_0. By the definition of a limit of a sequence, given $\varepsilon_1 > 0$ there exists an integer N, such that $|a_{n_0} - L| < \varepsilon_1$ for $n_0 > N$. Similarly, by the definition of continuity, the limit of f exists at L, so for $\varepsilon_2 > 0$ there exists $\varepsilon_1 > 0$ such that if $|a_{n_0} - L| < \varepsilon_1$ then $|f(a_{n_0}) - f(L)| < \varepsilon_2$. This is true for arbitrary $\varepsilon_2 > 0$, so $\lim_{n\to\infty} f(a_n) = f(L)$. □

Example 9. Find $\lim_{n\to\infty} \sin(\pi/n)$.

Example 10. Discuss the convergence of the sequence $a_n = n!/n^n$, where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.

Example 11. For what values of r is the sequence $\{r^n\}$ convergent?

Definition 11.1.5. A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is monotonic if it is either increasing or decreasing.

Example 12. Is the sequence $\left\{\dfrac{3}{n+5}\right\}$ increasing or decreasing?

Example 13. Show that the sequence $a_n = \dfrac{n}{n^2+1}$ is decreasing.

Definition 11.1.6. A sequence $\{a_n\}$ is bounded above if there is a number M such that

$$a_n \leq M \qquad \text{for all } n \geq 1.$$

It is bounded below if there is a number m such that

$$m \leq a_n \qquad \text{for all } n \geq 1.$$

If it is bounded above and below, then $\{a_n\}$ is a bounded sequence.

Theorem 11.1.6 (Monotonic Sequence theorem)**.** *Every bounded, monotonic sequence is convergent.*

Example 14. Investigate the sequence $\{a_n\}$ defined by the recurrence relation

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{2}(a_n + 6) \qquad \text{for } n = 1, 2, 3, \ldots.$$

11.2 Series

Definition 11.2.1. In general, if we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an infnite series (or just a series) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n.$$

Definition 11.2.2. Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} = s.$$

The number s is called the sum of the series. If the sequence $\{s_n\}$ is divergent, then the series is called divergent.

Example 1. Find the sum of the series $\sum_{n=1}^{\infty} a_n$ if the sum of the first n terms of the series is

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}.$$

Example 2. Find the sum of the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

where each term is obtained from the preceding one by multiplying it by the common ratio r.

Example 3. Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots .$$

Example 4. Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

Example 5. A drug is administered to a patient at the same time every day. Suppose the concentration of the drug is C_n (measured in mg/mL) after the injection on the nth day. Before the injection the next day, only 30% of the drug remains in the bloodstream and the daily dose raises the concentration by 0.2 mg/mL.

(a) Find the concentration after three days.

(b) What is the concentration after the nth dose?

(c) What is the limiting concentration?

Example 6. Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.

Example 7. Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$.

Example 8. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

Example 9. Show that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

Theorem 11.2.1. *If the series* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. Since $\sum a_n$ is convergent, the sequence $\{s_n\}$ is convergent. Let $\lim_{n\to\infty} s_n = s$. Since $n-1 \to \infty$ as $n \to \infty$, we also have $\lim_{n\to\infty} s_{n-1} = s$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0. \qquad \square$$

Corollary 11.2.1 (Test for Divergence)**.** *If* $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Proof. If the series is not divergent, then it is convergent, and so $\lim_{n\to\infty} a_n = 0$ by Theorem 11.2.1. The result follows by contrapositive. $\square$

Example 10. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Theorem 11.2.2. *If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$, and*

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Example 11. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.

Remark 1. A finite number of terms doesn't affect the convergence or divergence of a series. For instance, suppose that we were able to show that the series

$$\sum_{n=4}^{\infty}\frac{n}{n^3+1}$$

is convergent. Since

$$\sum_{n=1}^{\infty}\frac{n}{n^3+1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \sum_{n=4}^{\infty}\frac{n}{n^3+1}$$

it follows that the entire series $\sum_{n=1}^{\infty} n/(n^3+1)$ is convergent. Similarly, if it is known that the series $\sum_{n=N+1}^{\infty} a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.

11.3 The Integral Test and Estimates of Sums

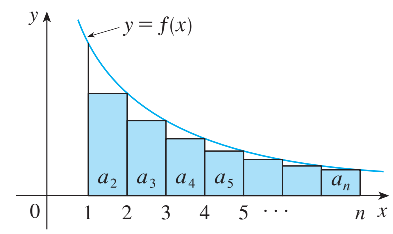
Theorem 11.3.1 (The Integral Test). *Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and $a_n = f(n)$. The the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:*

(i) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Proof.

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then comparing the areas of the rectangles with the area under $y = f(x)$ from 1 to n in the top figure, we see that



$$\sum_{i=2}^{n} a_i = a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\,dx \leq \int_1^{\infty} f(x)\,dx$$

since $f(x) \geq 0$. Therefore

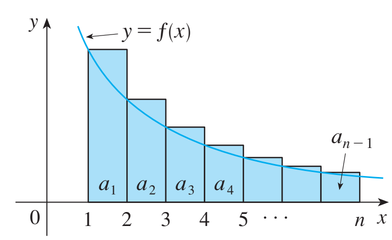
$$s_n = a_1 + \sum_{i=2}^{n} a_i \leq a_1 + \int_1^{\infty} f(x)\,dx = M, \text{ say.}$$

Since $s_n \leq M$ for all n, the sequence $\{s_n\}$ is bounded above. Also

$$s_{n+1} = s_n + a_{n+1} \geq s_n$$

since $a_{n+1} = f(n+1) \geq 0$. Thus $\{s_n\}$ is an increasing bounded sequence and so it is convergent by the Monotonic Sequence Theorem.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\int_1^n f(x)\,dx \to \infty$ as $n \to \infty$ because $f(x) \geq 0$. But the bottom figure shows that



$$\int_1^n f(x)\,dx \leq a_1 + a_2 + \cdots + a_{n-1} = \sum_{i=1}^{n-1} a_i = s_{n-1}$$

and so $s_{n-1} \to \infty$, implying that $s_n \to \infty$. □

Example 1. Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Example 2. For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent? (This series is called the p-series.)

Example 3. Determine whether each series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$

Example 4. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

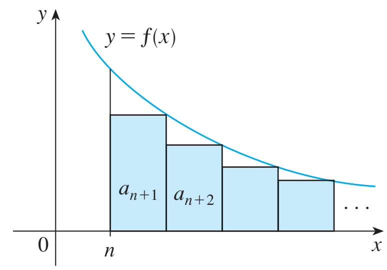
Definition 11.3.1. The remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

is the error made when s_n, the sum of the first n terms, is used as an approximation to the total sum.

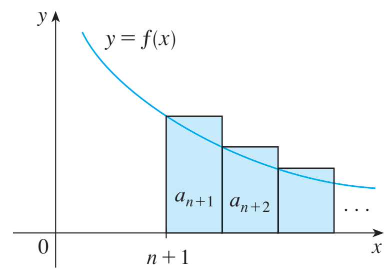
Theorem 11.3.2 (Remainder Estimate for the Integral Test)**.** *Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then*

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$



Proof. Comparing the rectangles with the area under $y = f(x)$ for $x > n$ in the top figure, we see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_{n}^{\infty} f(x)\,dx.$$



Similarly, we see from the bottom figure that

$$R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^{\infty} f(x)\,dx.$$

□

Example 5. (a) Approximate the sum of the series $\sum 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

Corollary 11.3.1. *Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. Then*

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\,dx.$$

Example 6. Use Corollary 11.3.1 with $n = 10$ to estimate the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

11.4 The Comparison Tests

Theorem 11.4.1 (The Comparison Test). *Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.*

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Proof. (i) Let

$$s_n = \sum_{i=1}^{n} a_i \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{\infty} b_n$$

Since both series have positive terms, the sequences $\{s_n\}$ and $\{t_n\}$ are increasing ($s_{n+1} = s_n + a_{n+1} \geq s_n$). Also $t_n \to t$, so $t_n \leq t$ for all n. Since $a_i \leq b_i$, we have $s_n \leq t_n$. Thus $s_n \leq t$ for all n. This means that $\{s_n\}$ is increasing and bounded above and therefore converges by the Monotonic Sequence Theorem. Thus $\sum a_n$ converges.

(ii) If $\sum b_n$ is divergent, then $t_n \to \infty$ (since $\{t_n\}$ is increasing). But $a_i \geq b_i$ so $s_n \geq t_n$. Thus $s_n \to \infty$. Therefore $\sum a_n$ diverges. □

Example 1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

Example 2. Test the series $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k}$ for convergence or divergence.

Theorem 11.4.2 (The Limit Comparison Test)**.** *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with positive terms. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c *is a finite number and* $c > 0$*, then either both series converge or both diverge.*

Proof. Let m and M be positive numbers such that $m < c < M$. Because a_n/b_n is close to c for large n, there is an integer N such that

$$m < \frac{a_n}{b_n} < M \qquad \text{when } n > N,$$

and so

$$mb_n < a_n < Mb_n \qquad \text{when } n > N.$$

If $\sum b_n$ converges, so does $\sum Mb_n$. Thus $\sum a_n$ converges by part (i) of the Comparison Test. If $\sum b_n$ diverges, so does $\sum mb_n$ and part (ii) of the Comparison Test shows that $\sum a_n$ diverges. □

Example 3. Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

Example 4. Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

Example 5. Use the sum of the first 100 terms to approximate the sum of the series $\sum 1/(n^3+1)$. Estimate the error involved in this approximation.

11.5 Alternating Series

Definition 11.5.1. An alternating series is a series whose terms are alternately positive and negative.

Theorem 11.5.1 (Alternating Series Test)**.** *If the alternating series*

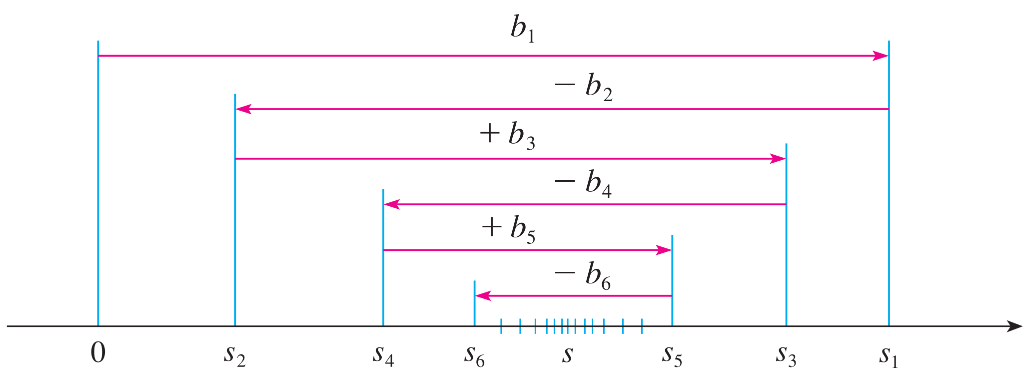
$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

$$\begin{aligned} &(i)\ \ b_{n+1} \le b_n \qquad \textit{for all } n \\ &(ii)\ \ \lim_{n\to\infty} b_n = 0 \end{aligned}$$

then the series is convergent.

Proof.



We first consider the even partial sums:

$$\begin{aligned} s_2 &= b_1 - b_2 \ge 0 && \text{since } b_2 \le b_1 \\ s_4 &= s_2 + (b_3 - b_4) \ge s_2 && \text{since } b_4 \le b_3. \end{aligned}$$

In general

$$s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \ge s_{2n-2} \qquad \text{since } b_{2n} \le b_{2n-1}.$$

Thus

$$0 \le s_2 \le s_4 \le s_6 \le \cdots \le s_{2n} \le \cdots.$$

But we can also write

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}.$$

Every term in parenthesis is positive, so $s_{2n} \leq b_1$ for all n. Therefore, the sequence $\{s_{2n}\}$ of even partial sums is increasing and bounded above. It is therefore convergent by the Monotonic Sequence Theorem. Let's call its limit s, that is,

$$\lim_{n\to\infty} s_{2n} = s.$$

Now we compute the limit of the odd partial sums:

$$\begin{aligned}\lim_{n\to\infty} s_{2n+1} &= \lim_{n\to\infty} (s_{2n} + b_{2n+1}) \\ &= \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} \\ &= s + 0 \\ &= s.\end{aligned}$$

Since both the even and odd partial sums converge to s, we have $\lim_{n\to\infty} s_n = s$ and so the series is convergent. $\square$

Example 1. Determine whether the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

converges or diverges.

Example 2. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ converges or diverges.

Example 3. Test the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ for convergence or divergence.

Theorem 11.5.2 (Alternating Series Estimation Theorem)**.** *If* $s = \sum(-1)^{n-1}b_n$, *where* $b_n > 0$, *is the sum of an alternating series that satisfies*

$$\text{(i)}\ \ b_{n+1} \leq b_n \qquad \text{and} \qquad \text{(ii)}\ \ \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Proof. We know from the proof of the Alternating Series Test that s lies between any two consecutive partial sums s_n and s_{n+1}. (There we showed that s is larger than all the even partial sums. A similar argument shows that s is smaller than all the odd sums.) It follows that

$$|s - s_n| \leq |s_{n+1} - s_n| = b_{n+1}. \qquad \square$$

Example 4. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

11.6 Absolute Convergence, Ratio and Root Tests

Definition 11.6.1. A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

Example 1. Is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

absolutely convergent?

Example 2. Is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

absolutely convergent?

Definition 11.6.2. A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

Theorem 11.6.1. *If a series $\sum a_n$ is absolutely convergent, then it is convergent.*

Proof. Observe that the inequality

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

is true because $|a_n|$ is either a_n or $-a_n$. If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent, so $\sum 2|a_n|$ is convergent. Therefore, by the Comparison Test, $\sum(a_n + |a_n|)$ is convergent. Then

$$\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$$

is the difference of two convergent series and is therefore convergent. $\square$

Example 3. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

Theorem 11.6.2 (The Ratio Test)**.**

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent (and therefore convergent).*

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ *or* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$*, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of* $\sum a_n$*.*

Example 4. Test the series $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ for absolute convergence.

Example 5. Test the convergence of the series $\sum_{n=1}^{\infty}\frac{n^n}{n!}$.

Theorem 11.6.3 (The Root Test)**.**

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent (and therefore convergent).*

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ *or* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$*, the Root Test is inconclusive.*

Example 6. Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

Definition 11.6.3. By a rearrangement of an infinite series $\sum a_n$ we mean a series obtained by simply changing the order of the terms.

Remark 1. If $\sum a_n$ is an absolutely convergent series with sum s, then any rearrangement of $\sum a_n$ has the same sum s.

Remark 2. If $\sum a_n$ is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\sum a_n$ that has a sum equal to r. For example, if we multiply the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2$$

by $\frac{1}{2}$, we get

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots = \frac{1}{2}\ln 2.$$

Then inserting zeros between the terms of this series gives

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots = \frac{1}{2}\ln 2,$$

and we can add this to the alternating harmonic series to get

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}\ln 2,$$

which is a rearrangement of the alternating harmonic series with a different sum.

11.7 Strategy for Testing Series

Example 1. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$.

Example 2. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$.

Example 3. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$.

Example 4. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{n^4+1}$.

Example 5. $\displaystyle\sum_{k=1}^{\infty}\frac{2^k}{k!}$.

Example 6. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2+3^n}$.

11.8 Power Series

Definition 11.8.1. A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.
More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a power series in $(x-a)$ or a power series centered at a or a power series about a.

Example 1. For what values of x is the series $\sum_{n=0}^{\infty} n! x^n$ convergent?

Example 2. For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Example 3. Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

Theorem 11.8.1. *For a given power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$, *there are only three possibilities:*

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$.

Definition 11.8.2. The number R in case (iii) is called the radius of convergence of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii). The interval of convergence of a power series is the interval that consists of all values of x for which the series converges.

Example 4. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}.$$

Example 5. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}.$$

11.9 Representations of Functions as Power Series

Example 1. Express $1/(1+x^2)$ as the sum of a power series and find the interval of convergence.

Example 2. Find a power series representation for $1/(x+2)$.

Example 3. Find a power series representation of $x^3/(x+2)$.

Theorem 11.9.1. *If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii)
$$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}.$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Example 4. Find the derivative of the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

Example 5. Express $1/(1-x)^2$ as a power series using differentiation. What is the radius of convergence?

Example 6. Find a power series representation for $\ln(1+x)$ and its radius of convergence.

Example 7. Find a power series representation for $f(x) = \tan^{-1} x$.

Example 8. (a) Evaluate $\int[1/(1+x^7)]dx$ as a power series.

(b) Use part (a) to approximate $\int_0^{0.5}[1/(1+x^7)]dx$ correct to within 10^{-7}.

11.10 Taylor and Maclaurin Series

Theorem 11.10.1. *If f has a power series representation (expansion) at a, that is, if*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Definition 11.10.1. The Taylor series of the function f at a (or about a or centered at a) is

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots . \end{aligned}$$

For the special case $a = 0$ the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots ,$$

which we call the Maclaurin Series.

Example 1. Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Theorem 11.10.2. *If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a, R_n is the remainder of the Taylor series, and*

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Theorem 11.10.3 (Taylor's Inequality)**.** *If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leq d.$$

Example 2. Prove that e^x is equal to the sum of its Maclaurin series.

Example 3. Find the Taylor series $f(x) = e^x$ at $a = 2$.

Example 4. Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

Example 5. Find the Maclaurin series for $\cos x$.

Example 6. Find the Maclaurin series for the function $f(x) = x \cos x$.

Example 7. Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\pi/3$.

Example 8. Find the Maclaurin series for $f(x) = (1 + x)^k$, where k is any real number.

Theorem 11.10.4 (The Binomial Series)**.** *If k is any real number and $|x| < 1$, then*

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

where the coefficients

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

are called the binomial coefficients.

Example 9. Find the Maclaurin series for the function $f(x) = \dfrac{1}{\sqrt{4-x}}$ and its radius of convergence.

Example 10. Find the sum of the series $\frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots$.

Example 11. (a) Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

(b) Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

Example 12. Evaluate $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$.

Example 13. Find the first three nonzero terms in the Maclaurin series for

(a) $e^x \sin x$

(b) $\tan x$

11.11 Applications of Taylor Polynomials

Example 1. (a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

(b) How accurate is this approximation when $7 \leq x \leq 9$?

Example 2. (a) What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when $-0.3 \leq x \leq 0.3$? Use this approximation to find $\sin 12^\circ$ correct to six decimal places.

(b) For what values of x is this approximation accurate to within 0.00005?

Example 3. In Einstein's theory of special relativity the mass of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of an object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0c^2.$$

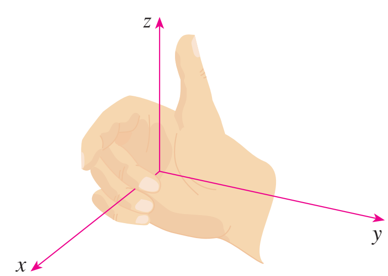
(a) Show that when v is very small compared with c, this expression for K agrees with classical Newtonian physics: $K = \frac{1}{2}m_0v^2$.

(b) Use Taylor's Inequality to estimate the difference in these expressions for K when $|v| \leq 100$ m/s.

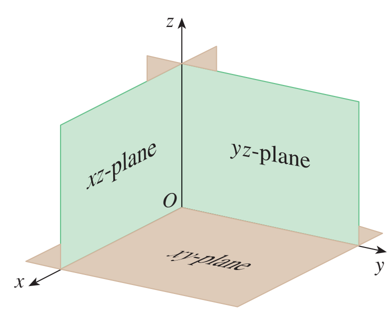
Chapter 12

Vectors and the Geometry of Space

12.1 Three-Dimensional Coordinate Systems



Definition 12.1.1. The coordinate axes are three directed lines through the origin that are perpendicular to each other and labeled the x-axis, y-axis, and z-axis. The direction of the z-axis is determined by the right-hand rule as illustrated in the figure.



Definition 12.1.2. The three coordinate axes determine the three coordinate planes illustrated in the figure. These three coordinate planes divide space into eight parts, called octants. The first octant, in the foreground of the figure, is determined by the positive axes.

Definition 12.1.3. We represent a point P in space by the ordered triple (a, b, c) where a is the distance from the yz-plane to P, b is the distance from the xz-plane to P, and c is the distance from the xy-plane to P. We call a, b, and c the coordinates of P. The points $(a, b, 0)$, $(0, b, c)$, and $(a, 0, c)$ are called the projections of P onto the xy-plane, yz-plane, and xz-plane, respectively.

Definition 12.1.4. The Cartesian product $\mathbb{R}\times\mathbb{R}\times\mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. It is called a three-dimensional rectangular coordinate system.

Example 1. What surfaces in $\mathbb{R}^3$ are represented by the following equations?

(a) $z = 3$

(b) $y = 5$

Example 2. (a) Which points (x, y, z) satisfy the equations

$$x^2 + y^2 = 1 \qquad \text{and} \qquad z = 3?$$

(b) What does the equation $x^2 + y^2 = 1$ represent as a surface in $\mathbb{R}^3$?

Example 3. Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $y = x$.

Theorem 12.1.1 (Distance Formula in Three Dimensions)**.** *The distance* $|P_1P_2|$ *between the points* $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$ *is*

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 4. Find the distance from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$.

Example 5. Find an equation of a sphere with radius r and center $C(h, k, l)$.

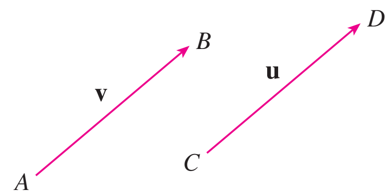
Example 6. Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

Example 7. What region in $\mathbb{R}^3$ is represented by the following inequalities?

$$1 \leq x^2 + y^2 + z^2 \leq 4 \qquad z \leq 0.$$

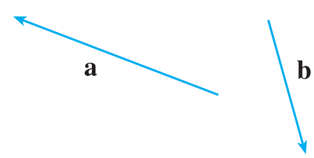
12.2 Vectors

Definition 12.2.1. A vector is a quantity that has both magnitude and direction, denoted $\mathbf{v}$ or $\vec{v}$. For a particle that moves along a line segment from point A to point B, the corresponding displacement vector, shown in the figure, has initial point A and terminal point B and we indicate this by writing $\mathbf{v} = \overrightarrow{AB}$. Because the vector $\mathbf{u} = \overrightarrow{CD}$ has the same length and the same direction as $\mathbf{v}$, even though it is in a different position, we say that $\mathbf{u}$ and $\mathbf{v}$ are equivalent (or equal) and we write $\mathbf{u} = \mathbf{v}$. The zero vector, denoted by $\mathbf{0}$ has length 0.



Definition 12.2.2 (Vector Addition)**.** If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the sum $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

Example 1. Draw the sum of the vectors $\mathbf{a}$ and $\mathbf{b}$ shown in the figure.



Definition 12.2.3 (Scalar Multiplication)**.** If c is a scalar and $\mathbf{v}$ is a vector, then the scalar multiple $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

Definition 12.2.4. Two nonzero vectors are parallel if they are scalar multiples of one another. In particular, the vector $-\mathbf{v} = (-1)\mathbf{v}$, called the negative of $\mathbf{v}$, has the same length as $\mathbf{v}$ but points in the opposite direction. By the difference $\mathbf{u} - \mathbf{v}$ of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

Example 2. If $\mathbf{a}$ and $\mathbf{b}$ are the vectors shown in the figure, draw $\mathbf{a} - 2\mathbf{b}$.

Definition 12.2.5. If we place the initial point of a vector $\mathbf{a}$ at the origin of a rectangular coordinate system, then the terminal point of $\mathbf{a}$ has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3). These coordinates are called the <u>components</u> of $\mathbf{a}$ and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \qquad \text{or} \qquad \mathbf{a} = \langle a_1, a_2, a_3 \rangle.$$

The <u>representation</u> of a vector from the origin to a point is called the <u>position vector</u> of the point.

Theorem 12.2.1. *Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is*

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

Proof. The vector $\mathbf{a} = \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ is the position vector of the point $P(a_1, a_2, a_3)$. If $\overrightarrow{AB}$ is another representation of $\mathbf{a}$, where the initial point is $A(x_1, y_1, z_1)$ and the terminal point is $B(x_2, y_2, z_2)$, then we must have $x_1 + a_1 = x_2$, $y_1 + a_2 = y_2$, and $z_1 + a_3 = z_2$. Therefore, $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, and $a_3 = z_2 - z_1$. □

Example 3. Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

Definition 12.2.6. The magnitude or length of the vector $\mathbf{v}$ is the length of any of its representations and is denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$.

Theorem 12.2.2. *The length of the two-dimensional vector* $\mathbf{a} = \langle a_1, a_2 \rangle$ *is*

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}.$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ *is*

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Theorem 12.2.3. *If* $\mathbf{a} = \langle a_1, a_2 \rangle$ *and* $\mathbf{b} = \langle b_1, b_2 \rangle$*, then*

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

and

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

for a scalar c. Similarly, for three-dimensional vectors,

$$\begin{aligned} \langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle &= \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle \\ \langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle &= \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle \\ c\langle a_1, a_2, a_3 \rangle &= \langle ca_1, ca_2, ca_3 \rangle. \end{aligned}$$

Example 4. If $\mathbf{a} = \langle 4, 0, 3 \rangle$ and $\mathbf{b} = \langle -2, 1, 5 \rangle$, find $|\mathbf{a}|$ and the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $3\mathbf{b}$, and $2\mathbf{a} + 5\mathbf{b}$.

Definition 12.2.7. We denote by V_2 the set of all two-dimensional vectors and by V_3 the set of all three-dimensional vectors. More generally, we denote by V_n the set of all n-dimensional vectors. An n-dimensional vector is an ordered n-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$$

where $a_1, a_2, \ldots, a_n$ are real numbers that are called the components of $\mathbf{a}$. Addition and scalar multiplication are defined in terms of components just as for the cases $n = 2$ and $n = 3$.

Theorem 12.2.4 (Properties of Vectors)**.** *If* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are vectors in* V_n *and* c *and* d *are scalars, then*

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$

4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$

7. $(cd)\mathbf{a} = c(d\mathbf{a})$

8. $1\mathbf{a} = \mathbf{a}$

Definition 12.2.8. The vectors

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

are called the standard basis vectors. They have length 1 and point in the directions of the positive x-, y-, and z-axes. Similarly, in two dimensions we define $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

Theorem 12.2.5. *Any vector in* V_3 *can be expressed in terms of* $\mathbf{i}$, $\mathbf{j}$, *and* $\mathbf{k}$. *Similarly, any vector in* V_2 *can be expressed in terms of* $\mathbf{i}$ *and* $\mathbf{j}$.

Proof. If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\begin{aligned} \mathbf{a} = \langle a_1, a_2, a_3 \rangle &= \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle \\ &= a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}. \end{aligned}$$

Similarly, in two dimensions, we can write

$$\mathbf{a} = \langle a_1, a_2 \rangle = a_1 \mathbf{i} + a_2 \mathbf{j}.$$ □

Example 5. If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 7\mathbf{k}$, express the vector $2\mathbf{a} + 3\mathbf{b}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Definition 12.2.9. A unit vector is a vector whose length is 1. For instance, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are all unit vectors.

Theorem 12.2.6. *In general, if* $\mathbf{a} \neq \mathbf{0}$*, then the unit vector that has the same direction as* $\mathbf{a}$ *is*

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

Proof. Let $c = 1/|\mathbf{a}|$. Then $\mathbf{u} = c\mathbf{a}$ and c is a positive scalar, so $\mathbf{u}$ has the same direction as $\mathbf{a}$. Also

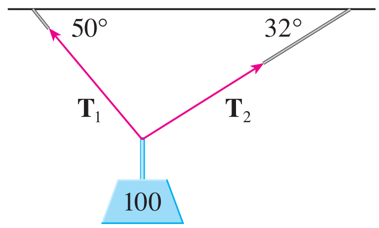
$$|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1.$$

□

Example 6. Find the unit vector in the direction of the vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

Definition 12.2.10. A force is represented by a vector because it has both a magnitude (measured in pounds or newtons) and a direction. If several forces are acting on an object, the resultant force experienced by the object is the vector sum of these forces.

Example 7. A 100-lb weight hangs from two wires as shown in the figure. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and the magnitudes of the tensions.



12.3 The Dot Product

Definition 12.3.1. If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the dot product of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and similarly

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

for two-dimensional vectors.

Example 1. Compute the following dot products:

(a) $\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle$

(b) $\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle$

(c) $(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k})$

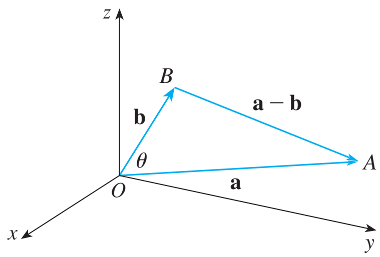
Theorem 12.3.1 (Properties of the Dot Product)**.** *If* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are vectors in* V_3 *and* c *is a scalar, then*

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot (\mathbf{b}) = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

Theorem 12.3.2. *If* θ *is the angle between the vectors* $\mathbf{a}$ *and* $\mathbf{b}$, *then*

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta.$$

Proof. If we apply the Law of Cosines to triangle OAB in the figure, we get



$$\begin{aligned}
|AB|^2 &= |OA|^2 + |OB|^2 - 2|OA||OB| \cos \theta \\
|\mathbf{a} - \mathbf{b}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta \\
(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta \\
\mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta \\
|\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta \\
-2\mathbf{a} \cdot \mathbf{b} &= -2|\mathbf{a}||\mathbf{b}| \cos \theta \\
\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos \theta
\end{aligned}$$

□

Example 2. If the vectors $\mathbf{a}$ and $\mathbf{b}$ have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

Corollary 12.3.1. *If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then*

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

Example 3. Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -3, 2 \rangle$.

Definition 12.3.2. Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are called perpendicular or orthogonal if the angle between them is $\theta = \pi/2$. The zero vector $\mathbf{0}$ is considered to be perpendicular to all vectors.

Theorem 12.3.3. *Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.*

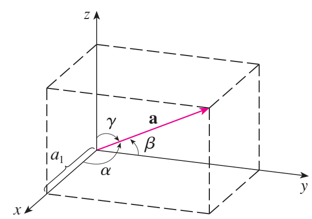
Proof. If $\theta = \pi/2$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/2) = 0.$$

Conversely, if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos\theta = 0$, so $\theta = \pi/2$. □

Example 4. Show that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

Definition 12.3.3. The direction angles of a nonzero vector $\mathbf{a}$ are the angles α, β, and γ (in the interval $[0, \pi]$) that $\mathbf{a}$ makes with the positive x-, y-, and z-axes, respectively. (See the figure.) The cosines of these direction angles, $\cos\alpha$, $\cos\beta$, $\cos\gamma$, are called the direction cosines of the vector $\mathbf{a}$.



Theorem 12.3.4. *The direction cosines of a vector* $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ *are the components of the unit vector in the direction of* $\mathbf{a}$*, i.e.,*

$$\frac{1}{|\mathbf{a}|}\mathbf{a} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle.$$

Proof. By Corollary 12.3.1,

$$\cos\alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}.$$

Similarly,

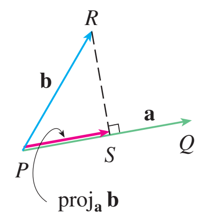
$$\cos\beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|}.$$

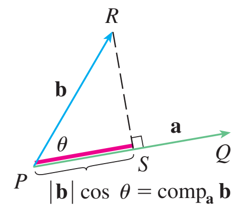
Therefore,

$$\begin{aligned} \mathbf{a} &= \langle a_1, a_2, a_3 \rangle \\ \mathbf{a} &= \langle |\mathbf{a}|\cos\alpha, |\mathbf{a}|\cos\beta, |\mathbf{a}|\cos\gamma \rangle \\ \mathbf{a} &= |\mathbf{a}|\langle \cos\alpha, \cos\beta, \cos\gamma \rangle \\ \frac{1}{|\mathbf{a}|}\mathbf{a} &= \langle \cos\alpha, \cos\beta, \cos\gamma \rangle. \end{aligned}$$ □

Example 5. Find the direction angles of the vector $\mathbf{a} = \langle 1, 2, 3 \rangle$.

Definition 12.3.4. If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ and is denoted by $\text{proj}_{\mathbf{a}}\,\mathbf{b}$. (See the figure.)
The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ (also called the component of $\mathbf{b}$ along $\mathbf{a}$) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}|\cos\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. (See the figure.) This is denoted by $\text{comp}_{\mathbf{a}}\,\mathbf{b}$.





Theorem 12.3.5. *The scalar projection of* $\mathbf{b}$ *onto* $\mathbf{a}$ *is*

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}.$$

The vector projection of $\mathbf{b}$ *onto* $\mathbf{a}$ *is*

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}.$$

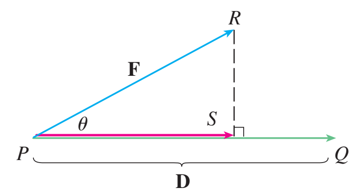
Proof. By Theorem 12.3.2,

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos\theta\\ \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} &= |\mathbf{b}|\cos\theta,\end{aligned}$$

which gives us the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$. Multiplying by the unit vector gives us the vector projection in the direction of $\mathbf{a}$. □

Example 6. Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2\rangle$ onto $\mathbf{a} = \langle -2, 3, 1\rangle$.

Definition 12.3.5. Suppose that the constant force in moving an object from P to Q is $\mathbf{F} = \overrightarrow{PR}$, as in the figure. Then the displacement vector is $\mathbf{D} = \overrightarrow{PQ}$ and the work done by this force is defined to be the product of the component of the force along $\mathbf{D}$ and the distance moved:



$$W = \big(|\mathbf{F}| \cos\theta\big)\, |\mathbf{D}|.$$

Theorem 12.3.6. *The work done by a constant force* $\mathbf{F}$ *is the dot product* $\mathbf{F} \cdot \mathbf{D}$*, where* $\mathbf{D}$ *is the displacement vector.*

Proof. By Theorem 12.3.2,

$$W = |\mathbf{F}||\mathbf{D}| \cos\theta = \mathbf{F} \cdot \mathbf{D}.$$ □

Example 7. A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal path. Find the work done by the force.

Example 8. A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.

12.4 The Cross Product

Definition 12.4.1. If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

Definition 12.4.2. A determinant of order 2 is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

A determinant of order 3 is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Theorem 12.4.1. *The cross product of the vectors* $\mathbf{a} = a_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ *and* $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ *is*

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}.$$

Example 1. If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, find $\mathbf{a} \times \mathbf{b}$.

Example 2. Show that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$ in V_3.

Theorem 12.4.2. *The vector* $\mathbf{a} \times \mathbf{b}$ *is orthogonal to both* $\mathbf{a}$ *and* $\mathbf{b}$.

Proof.

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\
&= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1) \\
&= a_1a_2b_3 - a_1b_2a_3 - a_1a_2b_3 + b_1a_2a_3 + a_1b_2a_3 - b_1a_2a_3 \\
&= 0.
\end{aligned}$$

Similarly, $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$. □

Theorem 12.4.3. *If* θ *is the angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *(so* $0 \le \theta \le \pi$*), then*

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta.$$

Proof.

$$\begin{aligned}
|\mathbf{a} \times \mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 \\
&\qquad + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta \\
&= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta) \\
&= |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta.
\end{aligned}$$

$\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \geq 0$ when $0 \leq \theta \leq \pi$, so

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta.$$ □

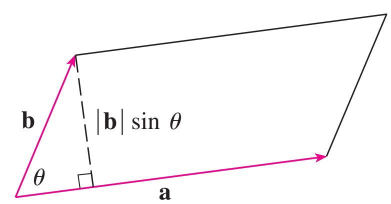
Corollary 12.4.1. *Two nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel if and only if*

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}.$$

Proof. Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\theta = 0$ or π. In either case $\sin\theta = 0$, so $|\mathbf{a} \times \mathbf{b}| = 0$ and therefore $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. □

Corollary 12.4.2. *The length of the cross product* $\mathbf{a} \times \mathbf{b}$ *is equal to the area of the parallelogram determined by* $\mathbf{a}$ *and* $\mathbf{b}$*.*

Proof. The geometric interpretation of Theorem 12.4.3. can be seen by looking at the figure. If $\mathbf{a}$ and $\mathbf{b}$ are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $|\mathbf{b}| \sin\theta$, and area



$$A = |\mathbf{a}|(|\mathbf{b}| \sin\theta) = |\mathbf{a} \times \mathbf{b}|.$$ □

Example 3. Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Example 4. Find the area of the triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.

Theorem 12.4.4. *If* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are vectors and* c *is a scalar, then*

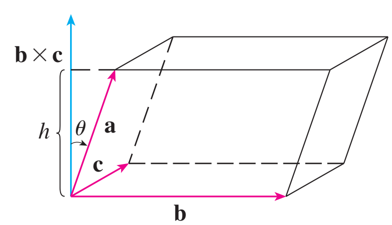
1. $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$
2. $(c\mathbf{a})\times\mathbf{b} = c(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(c\mathbf{b})$
3. $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$
4. $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$
5. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$
6. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

Theorem 12.4.5. *The volume of the parallelepiped determined by the vectors* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *is the magnitude of their scalar triple product:*

$$V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = \left\|\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}\right\|.$$

If the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *is 0, then the vectors must lie in the same plane; that is, they are coplanar.*

Proof. The geometric interpretation of the scalar triple product can be seen by looking at the figure. The area of the base parallelogram is $A = |\mathbf{b}\times\mathbf{c}|$. If θ is the angle between $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. Therefore the volume of the parallelepiped is



$$V = Ah = |\mathbf{b}\times\mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|. \qquad \square$$

Example 5. Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7\rangle$, $\mathbf{b} = \langle 2, -1, 4\rangle$, and $\mathbf{c} = \langle 0, -9, 18\rangle$ are coplanar.

Definition 12.4.3. If $\mathbf{F}$ is a force acting on a rigid body at a point given by a position vector $\mathbf{r}$ then the torque $\boldsymbol{\tau}$ (relative to the origin) is defined to be the cross product of the position and force vectors

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

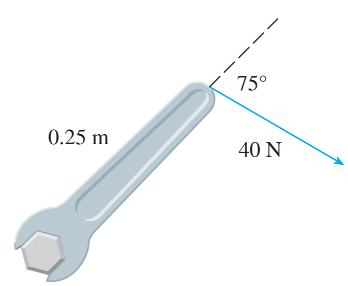
and measures the tendency of the body to rotate about the origin.

Theorem 12.4.6. *The magnitude of the torque vector is*

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin\theta$$

where θ is the angle between the position and force vectors.

Example 6. A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in the figure. Find the magnitude of the torque about the center of the bolt.

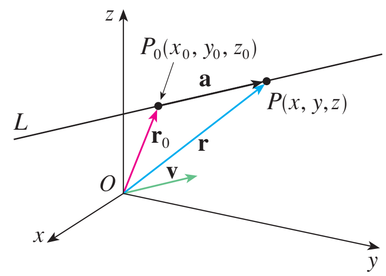


12.5 Equations of Lines and Planes

Theorem 12.5.1. *The vector equation of a line through the point* (x_0, y_0, z_0) *is*

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r}_0$ *is the position vector of* (x_0, y_0, z_0), $\mathbf{v}$ *is a vector parallel to the line, and* t *is a scalar.*
Parametric equations for a line through the point (x_0, y_0, z_0) *and parallel to the direction vector* $\langle a, b, c\rangle$ *are*



$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct.$$

Example 1. (a) Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

(b) Find two other points on the line.

Definition 12.5.1. In general, if a vector $\mathbf{v} = \langle a, b, c\rangle$ is used to describe the direction of a line L, then the numbers a, b, and c are called the direction numbers of L. The equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

obtained by eliminating the parameter t are called symmetric equations of L.

Example 2. (a) Find parametric equations and symmetric equations of the line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.

(b) At what point does this line intersect the xy-plane?

Theorem 12.5.2. *The line segment from* $\mathbf{r}_0$ *to* $\mathbf{r}_1$ *is given by the vector equation*

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \leq t \leq 1.$$

Example 3. Show that the lines L_1 and L_2 with parametric equations

$$L_1: \quad x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$
$$L_2: \quad x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

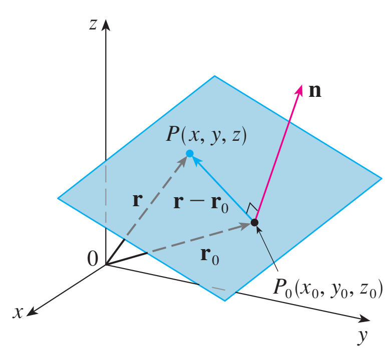
are skew lines; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

Definition 12.5.2. Either

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

or

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$



is called a vector equation of a plane through point (x_0, y_0, z_0) where $\mathbf{r}_0$ is the position vector of (x_0, y_0, z_0), $\mathbf{r}$ is the vector equation of the line through (x_0, y_0, z_0), and $\mathbf{n}$ is the vector through (x_0, y_0, z_0) orthogonal to the plane, called a normal vector.

A scalar equation of the plane through point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Example 4. Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$. Find the intercepts and sketch the plane.

Theorem 12.5.3. *The equation of a plane can be rewritten as the* *linear equation*

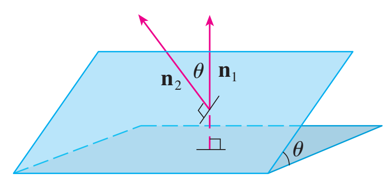
$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$.

Example 5. Find an equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

Example 6. Find the point at which the line with parametric equations $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

Definition 12.5.3. Two planes are parallel if their normal vectors are parallel. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors (see angle θ in the figure).



Example 7. (a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$

(b) Find symmetric equations for the line of intersection L of these two planes.

Example 8. Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

Example 9. Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

Example 10. In Example 3 we showed that the lines

$$\begin{array}{llll} L_1: & x = 1 + t & y = -2 + 3t & z = 4 - t \\ L_2: & x = 2s & y = 3 + s & z = -3 + 4s \end{array}$$

are skew. Find the distance between them.

12.6 Cylinders and Quadric Surfaces

Definition 12.6.1. The curves of intersection of a surface with planes parallel to the coordinate planes are called traces (or cross-sections) of the surface.

Definition 12.6.2. A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.

Example 1. Sketch the graph of the surface $z = x^2$.

Example 2. Identify and sketch the surfaces.

(a) $x^2 + y^2 = 1$

(b) $y^2 + z^2 = 1$

Definition 12.6.3. A quadric surface is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, B, C, \ldots, J$ are constants, but by translation and rotation it can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0.$$

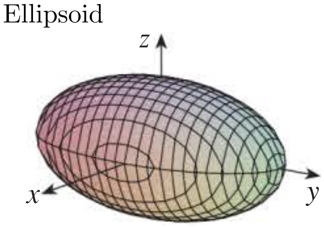
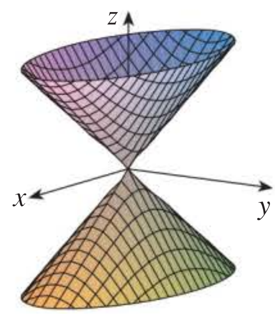
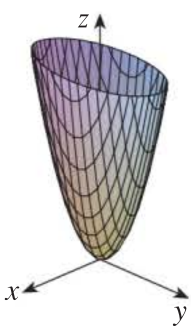
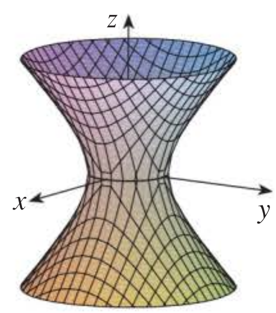
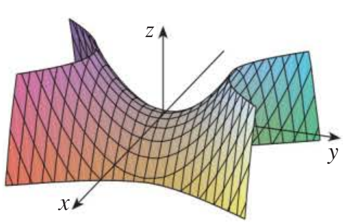
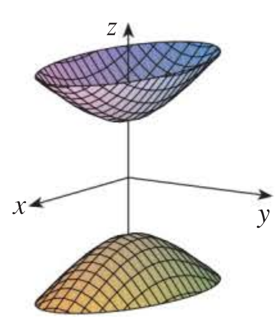
Example 3. Use traces to sketch the quadric surface with equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1.$$

Example 4. Use traces to sketch the surface $z = 4x^2 + y^2$.

Example 5. Sketch the surface $z = y^2 - x^2$.

Example 6. Sketch the surface $\dfrac{x^2}{4} + y^2 - \dfrac{z^2}{4} = 1$.

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Example 7. Identify and sketch the surface $4x^2 - y^2 + 2z^2 + 4 = 0$.

Example 8. Classify the quadric surface $x^2 + 2z^2 - 6x - y + 10 = 0$.

Chapter 13

Vector Functions

13.1 Vector Functions and Space Curves

Definition 13.1.1. A vector-valued function, or vector function is a function whose domain is a set of real numbers and whose range is a set of vectors. If $f(t)$, $g(t)$, and $h(t)$ are the components of a vector function $\mathbf{r}(t)$ whose values are three-dimensional vectors, then we call f, g, and h the component functions of $\mathbf{r}$ and we can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

Example 1. What are the component functions and domain of

$$\mathbf{r}(t) = \left\langle t^3, \ln(3-t), \sqrt{t} \right\rangle?$$

Definition 13.1.2. The limit of a vector function $\mathbf{r}$ is defined by taking the limits of its component functions, i.e., if $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

Example 2. Find $\lim_{t\to 0} \mathbf{r}(t)$, where $\mathbf{r}(t) = (1+t^3)\mathbf{i} + te^{-t}\mathbf{j} + \frac{\sin t}{t}\mathbf{k}$.

Definition 13.1.3. A vector function $\mathbf{r}$ is continuous at a if

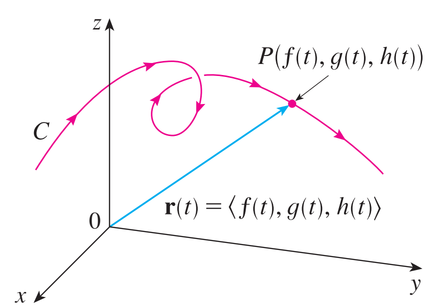
$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a),$$

so $\mathbf{r}$ is continuous at a if and only if its component functions f, g, and h are continuous at a.

Definition 13.1.4. Suppose that f, g, and h are continuous real-valued functions on an interval I. Then the set C of all points (x, y, z) in space, where

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

(called the parametric equations of C for a parameter t) and t varies throughout the interval I, is called a space curve. If we consider the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then $\mathbf{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on C. Thus any continuous vector function $\mathbf{r}$ defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$, as shown in the figure.



Example 3. Describe the curve defined by the vector function

$$\mathbf{r}(t) = \langle 1+t, 2+5t, -1+6t\rangle.$$

Example 4. Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}.$$

Example 5. Find a vector equation and parametric equations for the line segment that joins the point $P(1, 3, -2)$ to the point $Q(2, -1, 3)$.

Example 6. Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

Example 7. Use a computer to draw the curve with vector equation $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$. This curve is called a twisted cubic.

13.2 Vector Function Derivatives & Integrals

Definition 13.2.1. The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined as

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists.

Definition 13.2.2. The vector $\mathbf{r}'(t)$ is called the tangent vector to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$. The tangent line to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Theorem 13.2.1. *If* $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, *where* f, g, *and* h *are differentiable functions, then*

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Proof.

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}[\mathbf{r}(t+\Delta t) - \mathbf{r}(t)] \\
&= \lim_{\Delta t\to 0} \frac{1}{\Delta t}[\langle f(t+\Delta t), g(t+\Delta t), h(t+\Delta t)\rangle - \langle f(t), g(t), h(t)\rangle] \\
&= \lim_{\Delta t\to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle \\
&= \left\langle \lim_{\Delta t\to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}, \lim_{\Delta t\to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}, \lim_{\Delta t\to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle \\
&= \langle f'(t), g'(t), h'(t)\rangle.
\end{aligned}$$

□

Example 1. (a) Find the derivative of $\mathbf{r}(t) = (1+t^3)\mathbf{i} + te^{-t}\mathbf{j} + \sin 2t\mathbf{k}$.

(b) Find the unit tangent vector at the point where $t = 0$.

Example 2. For the curve $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (2 - t)\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$.

Example 3. Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $(0, 1, \pi/2)$.

Definition 13.2.3. The <u>second derivative</u> of a vector function $\mathbf{r}$ is the derivative of $\mathbf{r}'$, that is, $\mathbf{r}'' = (\mathbf{r}')'$.

Theorem 13.2.2. *Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are differentiable vector functions,* c *is a scalar, and* f *is a real-valued function. Then*

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Example 4. Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

Definition 13.2.4. The definite integral of a continuous vector function $\mathbf{r}(t)$ is

$$\begin{aligned}\int_a^b \mathbf{r}(t)dt &= \lim_{n\to\infty}\sum_{i=1}^{n}\mathbf{r}(t_i^*)\Delta t\\ &= \lim_{n\to\infty}\left[\left(\sum_{i=1}^{n}f(t_i^*)\Delta t\right)\mathbf{i}+\left(\sum_{i=1}^{n}g(t_i^*)\Delta t\right)\mathbf{j}+\left(\sum_{i=1}^{n}h(t_i^*)\Delta t\right)\mathbf{k}\right]\end{aligned}$$

and so

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\mathbf{i}+\left(\int_a^b g(t)dt\right)\mathbf{j}+\left(\int_a^b h(t)dt\right)\mathbf{k}.$$

Theorem 13.2.3. *We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:*

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a).$$

where $\mathbf{R}$ *is an antiderivative of* $\mathbf{r}$*, that is,* $\mathbf{R}'(t) = \mathbf{r}(t)$*. We use the notation* $\int \mathbf{r}(t)dt$ *for indefinite integrals (antiderivatives).*

Example 5. If $\mathbf{r}(t) = 2\cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$, then what are $\int \mathbf{r}(t)dt$ and $\int_0^{\pi/2}\mathbf{r}(t)dt$?

13.3 Arc Length and Curvature

Definition 13.3.1. If a space curve is given by $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, $a \leq t \leq b$, or equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f', g', and h' are continuous, then the length of the curve traversed exactly once as t increases from a to b is

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt, \end{aligned}$$

or equivalently,

$$L = \int_a^b |\mathbf{r}'(t)|dt.$$

Example 1. Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

Remark 1. A single curve C can be represented by more than one vector function. For instance, the twisted cubic

$$\mathbf{r}_1(t) = \langle t, t^2, t^3\rangle \qquad 1 \leq t \leq 2$$

could also be represented by the function

$$\mathbf{r}_2(u) = \langle e^u, e^{2u}, e^{3u}\rangle \qquad 0 \leq u \leq \ln 2$$

We say that these equations are parametrizations of the curve C. It can be shown that our arc length equation is independent of the parametrization that is used.

Definition 13.3.2. Suppose that C is a curve given by a vector function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \qquad a \leq t \leq b$$

where $\mathbf{r}'$ is continuous and C is traversed exactly once as t increases from a to b. We define its arc length function s by

$$s(t) = \int_a^t |\mathbf{r}'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du$$

where differentiating both sides of the arc length function using the Fundamental Theorem of Calculus gives

$$\frac{ds}{dt} = |\mathbf{r}'(t)|.$$

Remark 2. If a curve $\mathbf{r}(t)$ is already given in terms of a parameter t and $s(t)$ is the arc length function, then we may be able to solve for t as a function of s: $t = t(s)$. Then the curve can be reparametrized with respect to arc length by substituting for t: $\mathbf{r} = \mathbf{r}(t(s))$.

Example 2. Reparametrize the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

Definition 13.3.3. The curvature of a curve C at a given point is a measure of how quickly the curve changes direction at that point, defined as

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector.

Remark 3. A parametrization is called smooth on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on I. A curve is called smooth if it has a smooth parametrization. Since the unit tangent vector is only defined for smooth curves, the curvature is only defined for smooth curves.

Theorem 13.3.1.

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

Proof. By the chain rule

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt},$$

so

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}/dt}{ds/dt} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

□

Example 3. Show that the curvature of a circle of radius a is $1/a$.

Theorem 13.3.2. *The curvature of the curve given by the vector function* $\mathbf{r}$ *is*

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Proof. Since $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ and $|\mathbf{r}'| = ds/dt$, we have

$$\mathbf{r}' = |\mathbf{r}'|\mathbf{T} = \frac{ds}{dt}\mathbf{T}$$
$$\mathbf{r}'' = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'.$$

Since $\mathbf{T} \times \mathbf{T} = \mathbf{0}$, we have

$$\mathbf{r}' \times \mathbf{r}'' = \frac{ds}{dt}\mathbf{T} \times \left(\frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'\right)$$
$$\mathbf{r}' \times \mathbf{r}'' = \frac{ds}{dt}\mathbf{T} \times \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T} \times \frac{ds}{dt}\mathbf{T}'$$
$$\mathbf{r}' \times \mathbf{r}'' = \left(\frac{ds}{dt}\frac{d^2s}{dt^2}\right)(\mathbf{T} \times \mathbf{T}) + \left(\frac{ds}{dt}\right)^2 (\mathbf{T} \times \mathbf{T}')$$
$$\mathbf{r}' \times \mathbf{r}'' = \left(\frac{ds}{dt}\right)^2 (\mathbf{T} \times \mathbf{T}').$$

Since $|\mathbf{T}(t)| = 1$ for all t, $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal, so

$$\begin{aligned} |\mathbf{r}' \times \mathbf{r}''| &= \left(\frac{ds}{dt}\right)^2 |\mathbf{T} \times \mathbf{T}'| \\ &= \left(\frac{ds}{dt}\right)^2 |\mathbf{T}||\mathbf{T}'| \sin\left(\frac{\pi}{2}\right) \\ &= \left(\frac{ds}{dt}\right)^2 |\mathbf{T}'|. \end{aligned}$$

Thus

$$|\mathbf{T}'| = \frac{|\mathbf{r}' \times \mathbf{r}''|}{(ds/dt)^2} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^2}$$

and

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}.$$

□

Example 4. Find the curvature of the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at a general point and at $(0, 0, 0)$.

Theorem 13.3.3. *If $y = f(x)$ is a plane curve, then*

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

Proof. Choose x as the parameter and write $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j}$. Then $\mathbf{r}'(x) = \mathbf{i} + f'(x)\mathbf{j}$ and $\mathbf{r}''(x) = f''(x)\mathbf{j}$. Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, it follows that $\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x)\mathbf{k}$. We also have $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$, and so

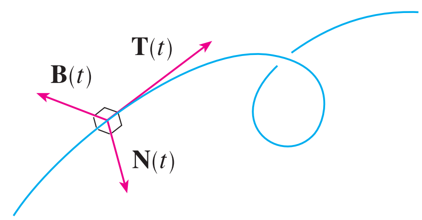
$$\kappa(x) = \frac{|\mathbf{r}'(x) \times \mathbf{r}''(x)|}{|\mathbf{r}'(x)|^3} = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$ □

Example 5. Find the curvature of the parabola $y = x^2$ at the points $(0, 0)$, $(1, 1)$, and $(2, 4)$.

Definition 13.3.4. For any point where $\kappa \neq 0$, the principal unit normal vector $\mathbf{N}(t)$ (or simply unit normal) is defined to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|},$$

and so it is orthogonal to the unit tangent vector $\mathbf{T}(t)$. The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the binormal vector. It is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and is also a unit vector. (See the figure.)



Example 6. Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}.$$

Definition 13.3.5. The plane determined by the normal and binormal vectors **N** and **B** at point P on a curve C is called the normal plane of C at P. It consists of all lines that are orthogonal to the tangent vector **T**. The plane determined by the vectors **T** and **N** is called the osculating plane of C at P. It is the plane that comes closest to containing the part of the curve near P.

Definition 13.3.6. The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which **N** points), and has radius $\rho = 1/\kappa$ (the reciprocal of the curvature) is called the osculating circle (or the circle of curvature) of C at P. It is the circle that best describes how C behaves near P; it shares the same tangent, normal, and curvature at P.

Example 7. Find equations of the normal plane and osculating plane of the helix in Example 6 at the point $P(0, 1, \pi/2)$.

Example 8. Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

13.4 Motion in Space

Definition 13.4.1. Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$. Then the velocity vector $\mathbf{v}(t)$ at time t is given by

$$\mathbf{v}(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t).$$

The speed of the particle at time t is the magnitude of the velocity vector, that is, $|\mathbf{v}(t)|$. As in the case of one-dimensional motion, the acceleration of the particle is defined as the derivative of the velocity:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).$$

Example 1. The position vector of an object moving in a plane is given by $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$. Find its velocity, speed, and acceleration when $t = 1$ and illustrate geometrically.

Example 2. Find the velocity, acceleration, and speed of a particle with position vector $\mathbf{r}(t) = \langle t^2, e^t, te^t \rangle$.

Example 3. A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 4t\mathbf{i} + 6t\mathbf{j} + \mathbf{k}$. Find its velocity and position at time t.

Remark 1. In general, vector integrals allow us to recover velocity when acceleration is known and position when velocity is known:

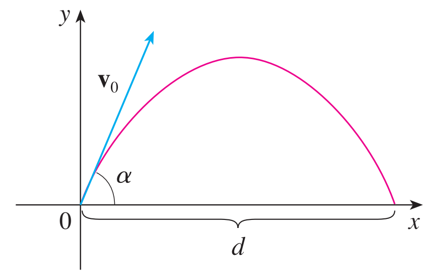
$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(u)du \qquad \mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(u)du.$$

If the force that acts on a particle is known, then the acceleration can be found from Newton's Second Law of Motion. The vector version of this law states that if, at any time t, a force $\mathbf{F}(t)$ acts on an object of mass m producing an acceleration $\mathbf{a}(t)$, then

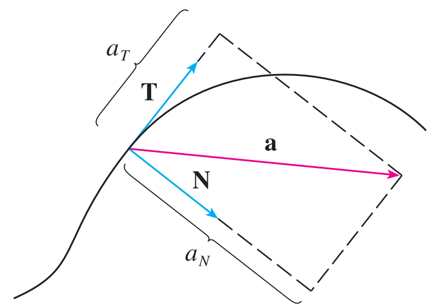
$$\mathbf{F}(t) = m\mathbf{a}(t).$$

Example 4. An object with mass m that moves in a circular path with constant angular speed ω has position vector $\mathbf{r}(t) = a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j}$. Find the force acting on the object and show that it is directed toward the origin.

Example 5. A projectile is fired with angle of elevation α and initial velocity $\mathbf{v}_0$. (See the figure.) Assuming that air resistance is negligible and the only external force is due to gravity, find the position function $\mathbf{r}(t)$ of the projectile. What value of α maximizes the range (the horizontal distance traveled)?



Example 6. A projectile is fired with muzzle speed 150 m/s and angle of elevation $45°$ from a position 10 m above ground level. Where does the projectile hit the ground, and with what speed?



Theorem 13.4.1. *If $v = |\mathbf{v}|$ is the speed of a particle in motion, then*

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

where $a_T = v'$ and $a_N = \kappa v^2$.

Proof.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{v}}{v},$$

so

$$\mathbf{v} = v\mathbf{T}$$
$$\mathbf{a} = \mathbf{v}' = v'\mathbf{T} + v\mathbf{T}'.$$

By our expression for curvature,

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v},$$

so $|\mathbf{T}'| = \kappa v$. Since $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$,

$$\mathbf{T}' = |\mathbf{T}'|\mathbf{N} = \kappa v\mathbf{N},$$

and thus

$$\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$$ □

Theorem 13.4.2.

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} \qquad a_N = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}.$$

Proof.

$$\begin{aligned}\mathbf{v}\cdot\mathbf{a} &= v\mathbf{T}\cdot(v'\mathbf{T} + \kappa v^2\mathbf{N})\\ &= vv'\mathbf{T}\cdot\mathbf{T} + \kappa v^3\mathbf{T}\cdot\mathbf{N}\\ &= vv',\end{aligned}$$

so

$$\begin{aligned}a_T &= v' = \frac{\mathbf{v}\cdot\mathbf{a}}{v} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}\\ a_N &= \kappa v^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}|\mathbf{r}'(t)|^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}.\end{aligned}$$ □

Example 7. A particle moves with position function $\mathbf{r}(t) = \langle t^2, t^2, t^3 \rangle$. Find the tangential and normal components of acceleration.

Chapter 14

Partial Derivatives

14.1 Functions of Several Variables

Definition 14.1.1. A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

Remark 1. We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are independent variables and z is the dependent variable.

Example 1. For each of the following functions, evaluate $f(3, 2)$ and find and sketch the domain.

(a) $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$

(b) $f(x, y) = x \ln(y^2 - x)$

Example 2. In regions with severe winter weather, the *wind-chill index* is often used to describe the apparent severity of the cold. This index W is a subjective temperature that depends on the actual temperature T and the wind speed v. So W is a function of T and v, and we can write $W = f(T, v)$. The table records values of W compiled by the US National Weather Service and the Meteorological Service of Canada.

Wind-chill index as a function of air temperature and wind speed

Wind speed (km/h)

Actual temperature (°C)

T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

Find $f(-5, 50)$ and interpret its meaning in context.

Example 3. In 1928 Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899-1922. They considered a simplified view of the economy in which production output is determined by the amount of labor involved and the amount of capital invested. While there are many other factors affecting economic performance, their model proved to be remarkably accurate. The function they used to model production was of the form

$$P(L, K) = bL^{\alpha}K^{1-\alpha},$$

known as the Cobb-Douglas production function, where P is the total production (the monetary value of all goods produced in a year), L is the amount of labor (the total number of person-hours worked in a year), and K is the amount of capital invested (the monetary worth of all machinery, equipment, and buildings).
Cobb and Douglas used economic data published by the government to obtain the table on the right. They took the year 1899 as a baseline and P, L, and K for 1899 were each assigned the value 100. The values for other years were expressed as percentages of the 1899 figures.
Cobb and Douglas used the method of least squares to fit the data of the table to the function

$$P(L, K) = 1.01L^{0.75}K^{0.25}.$$

Use this function to compute the production in the years 1910 and 1920, and compare your results with the actual values for these years.

Year	P	L	K
1899	100	100	100
1900	101	105	107
1901	112	110	114
1902	122	117	122
1903	124	122	131
1904	122	121	138
1905	143	125	149
1906	152	134	163
1907	151	140	176
1908	126	123	185
1909	155	143	198
1910	159	147	208
1911	153	148	216
1912	177	155	226
1913	184	156	236
1914	169	152	244
1915	189	156	246
1916	225	183	298
1917	227	198	335
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

Example 4. Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Definition 14.1.2. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Definition 14.1.3. The level curves of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

Example 5. Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

Definition 14.1.4. The function

$$f(x, y) = ax + by + c$$

is called a linear function. The graph of such a function has the equation

$$z = ax + by + c \qquad \text{or} \qquad ax + by - z + c = 0,$$

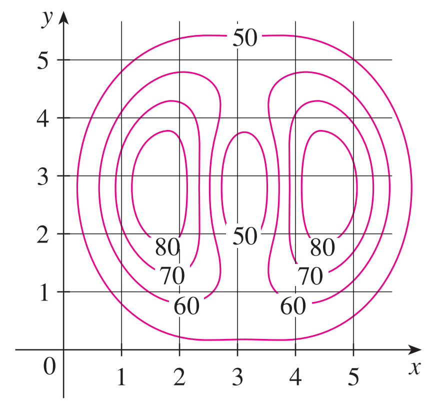
so it is a plane.

Example 6. Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Example 7. Use a computer to draw the graph of the Cobb-Douglas production function $P(L, K) = 1.01L^{0.75}K^{0.25}$.

Example 8. Find the domain and range and sketch the graph of $h(x, y) = 4x^2 + y^2$.

Example 9. A contour map for a function f is shown in the figure. Use it to estimate the values of $f(1,3)$ and $f(4,5)$.



Example 10. Sketch the level curves of the function $f(x,y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

Example 11. Sketch the level curves of the function

$$g(x,y) = \sqrt{9 - x^2 - y^2} \qquad \text{for} \quad k = 0, 1, 2, 3.$$

Example 12. Sketch some level curves of the function $h(x, y) = 4x^2 + y^2 + 1$.

Example 13. Plot level curves for the Cobb-Douglas production function of Example 3.

Definition 14.1.5. A function of three variables, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.

Example 14. Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z.$$

Definition 14.1.6. The level surfaces of a function f of three variables are the curves with equations $f(x, y, z) = k$, where k is a constant.

Example 15. Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2.$$

Definition 14.1.7. A function of n variables is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples.

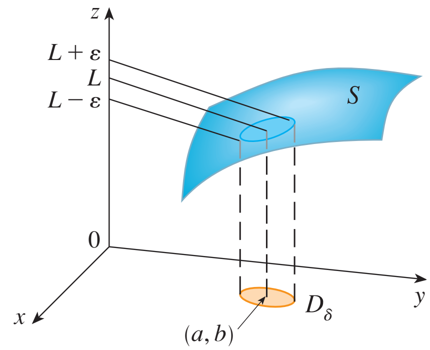
Remark 2. Sometimes we will use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we will often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \ldots, x_n)$.

14.2 Limits and Continuity

Definition 14.2.1. Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \varepsilon$.



Remark 1. If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Example 1. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Example 2. If $f(x, y) = xy/(x^2 + y^2)$, does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

Example 3. If $f(x,y) = \dfrac{xy^2}{x^2+y^4}$, does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

Remark 2. The Limit Laws listed in section 2.3 can be extended to functions of two variables: the limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and so on. In particular, the following equations are true.

$$\lim_{(x,y)\to(a,b)} x = a \qquad \lim_{(x,y)\to(a,b)} y = b \qquad \lim_{(x,y)\to(a,b)} c = c.$$

The Squeeze Theorem also holds.

Example 4. Find $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

Definition 14.2.2. A function f of two variables is called continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

We say that f is continuous on D if f is continuous at every point (a, b) in D.

Definition 14.2.3. A polynomial of two variables (or polynomial, for short) is a sum of terms of the form cx^my^n, where c is a constant and m and n are nonnegative integers. A rational function is a ratio of polynomials.

Remark 3. The limits in Remark 2 show that the functions $f(x, y) = x$, $g(x, y) = y$, and $h(x, y) = c$ are continuous. Since any polynomial can be built up out of the simple functions f, g, and h by multiplication and addition, it follows that all polynomials are continuous on $\mathbb{R}^2$. Likewise, any rational function is continuous on its domain because it is a quotient of continuous functions.

Example 5. Evaluate $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

Example 6. Where is the function $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

Example 7. Where is the function

$$g(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous?

Remark 4. If f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x,y) = g(f(x,y))$ is also a continuous function.

Example 8. Where is the function

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous?

Example 9. Where is the function $h(x,y) = \arctan(y/x)$ continuous?

Definition 14.2.4. The notation

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$$

means that the values of $f(x,y,z)$ approach the number L as the point (x,y,z) approaches the point (a,b,c) along any path in the domain of f. Precisely, for every number $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that if $f(x,y,z)$ is in the domain of f and $0 < \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} < \delta$ then $|f(x,y,z) - L| < \varepsilon$. The function is <u>continuous</u> at (a,b,c) if

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c).$$

Definition 14.2.5. If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $\mathbf{x} \in D$ and $0 < |\mathbf{x} - \mathbf{a}| < \delta$ then $|f(\mathbf{x}) - L| < \varepsilon$. The function is continuous at $\mathbf{a}$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).$$

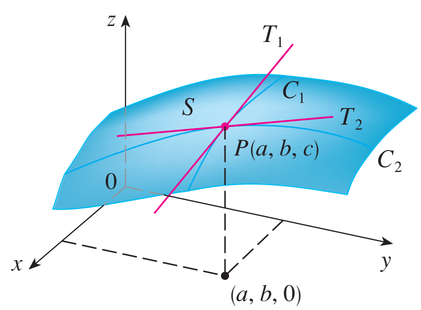
14.3 Partial Derivatives

Definition 14.3.1. In general, if f is a function of two variables x and y, suppose we only let x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are considering a function of a single variable x, say $g(x) = f(x, b)$. If g has a derivative at a, then we call it the partial derivative of f with respect to x at (a, b) and denote it by $f_x(a, b)$. Thus

$$f_x(a,b) = g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

Similarly, the partial derivative of f with respect to y at (a, b), denoted by $f_y(a, b)$, is obtained by keeping x fixed ($x = a$) and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$



Definition 14.3.2. If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$
$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Definition 14.3.3 (Notations for Partial Derivatives)**.** If $z = f(x, y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f.$$

Remark 1 (Rule for Finding Partial Derivatives of $z = f(x, y)$).

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example 1. If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Example 2. If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

Example 3. The body mass index of a person is defined by

$$B(m, h) = \frac{m}{h^2}.$$

Calculate the partial derivatives of B for a young man with $m = 64$ kg and $h = 1.68$ m and interpret them.

Example 4. If $f(x, y) = \sin\left(\dfrac{x}{1+y}\right)$, calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Example 5. Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1.$$

Definition 14.3.4. If f is a function of three variables x, y and z, then its partial derivative with respect to x is defined as

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding y and z as constants and differentiating $f(x, y, z)$ with respect to x.

Definition 14.3.5. In general, if u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the ith variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f.$$

Example 6. Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$.

Definition 14.3.6. If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the second partial derivatives of f. If $z = f(x, y)$, we use the following notation:

$$\begin{aligned}
(f_x)_x &= f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} \\
(f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} \\
(f_y)_x &= f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} \\
(f_y)_y &= f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}.
\end{aligned}$$

Thus the notation f_{xy} (or $\partial^2 f/\partial y \partial x$) means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

Example 7. Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2.$$

Theorem 14.3.1 (Clairaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Remark 2. Partial derivatives of order 3 or higher can also be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y^2 \partial x}$$

and using Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

Example 8. Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

Definition 14.3.7. The partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called Laplace's equation. Solutions of this equation are called harmonic functions.

Example 9. Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

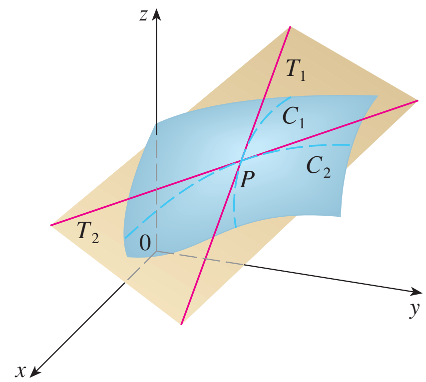
Definition 14.3.8. The wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibration string.

Example 10. Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

14.4 Tangent Planes & Linear Approximations



Definition 14.4.1. Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S, so that P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. Then the tangent plane to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2. (See the figure.)

Theorem 14.4.1. *Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is*

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Proof. Any line passing through P has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

By dividing this equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

$$z - z_0 = a(x - x_0) + b(y - y_0).$$

If this equation represents the tangent plane at P, then its intersection with the tangent line $y = y_0$ must be T_1, so by letting $y = y_0$ we get

$$z - z_0 = a(x - x_0)$$

as the equation of T_1, and since T_1 has slope $f_x(x_0, y_0)$, we have $a = f_x(x_0, y_0)$. Similarly, by letting $x = x_0$, we get $z - z_0 = b(y - y_0)$ as the equation of T_2, so $b = f_y(x_0, y_0)$. □

Example 1. Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Definition 14.4.2. The linear function whose graph is the tangent plane at the point to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is called the linearization of f at (a, b) and is given by

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linear approximation or the tangent line approximation of f at (a, b).

Definition 14.4.3. Suppose $z = f(x, y)$ is a function of two variables where x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. Then the corresponding increment of z is

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b).$$

Definition 14.4.4. If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Theorem 14.4.2. *If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).*

Example 2. Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Example 3. On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates. The National Weather Service has devised the *heat index* (also called the temperature-humidity index, or humidex, in some countries) to describe the combined effects of temperature and humidity. The heat index I is the perceived air temperature when the actual temperature is T and the relative humidity is H. So I is a function of T and H and we can write $I = f(T, H)$. The following table of values of I is an excerpt from a table compiled by the National Weather Service.

Heat index I as a function of temperature and humidity

Relative humidity (%)

Actual temperature (°F)

T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Find a linear approximation for the heat index $I = f(T, H)$ when T is near 96°F and H is near 70%. Use it to estimate the heat index when the temperature is 97°F and the relative humidity is 72%.

Definition 14.4.5. For a differentiable function of two variables, $z = f(x, y)$, we define the differentials dx and dy to be independent variables; that is, they can be given any values. Then the differential dz, also called the total differential, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Example 4.

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

Example 5. The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

Remark 1. Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables. A differentiable function is defined by an expression similar to the one in Definition 14.4.4. For such functions the linear approximation is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and the linearization $L(x, y, z)$ is the right side of this expression.
If $w = f(x, y, z)$ then the increment of w is

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z).$$

The differential dw is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz.$$

Example 6. The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

14.5 The Chain Rule

Theorem 14.5.1 (The Chain Rule (Case 1))**.** *Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are differentiable functions of t. Then z is a differentiable function of t and*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Proof.

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Dividing both sides of this equation by Δt, we have

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}.$$

If we now let $\Delta t \to 0$, then $\Delta x = g(t+\Delta t) - g(t) \to 0$ because g is differentiable and therefore continuous. Similarly, $\Delta y \to 0$. This, in turn, means that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, so

$$\begin{aligned}
\frac{dz}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} \\
&= \frac{\partial f}{\partial x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} + \left(\lim_{\Delta t \to 0} \varepsilon_1\right) \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \left(\lim_{\Delta t \to 0} \varepsilon_2\right) \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt} \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.
\end{aligned}$$

□

Example 1. If $z = x^2 y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

Example 2. The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

Theorem 14.5.2 (The Chain Rule (Case 2))**.** *Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Example 3. If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

Theorem 14.5.3 (The Chain Rule (General Version))**.** *Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and*

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

Example 4. Write out the Chain Rule for the case where $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$.

Example 5. If $u = x^4 y + y^2 z^3$, where $x = rse^t$, $y = rs^2 e^{-t}$, and $z = r^2 s \sin t$, find the value of $\partial u / \partial s$ when $r = 2$, $s = 1$, $t = 0$.

Example 6. If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

Example 7. If $z = f(x,y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find

(a) $\partial z/\partial r$

(b) $\partial^2 z/\partial r^2$

Theorem 14.5.4 (Implicit Differentiation)**.** *Suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f. If F is differentiable,*

$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}.$$

Proof. If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x to get

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

But $dx/dx = 1$, so if $\partial F/\partial y \neq 0$ we can solve for dy/dx and obtain the desired result. □

Example 8. Find y' if $x^3 + y^3 = 6xy$.

Theorem 14.5.5. *Suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable,*

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

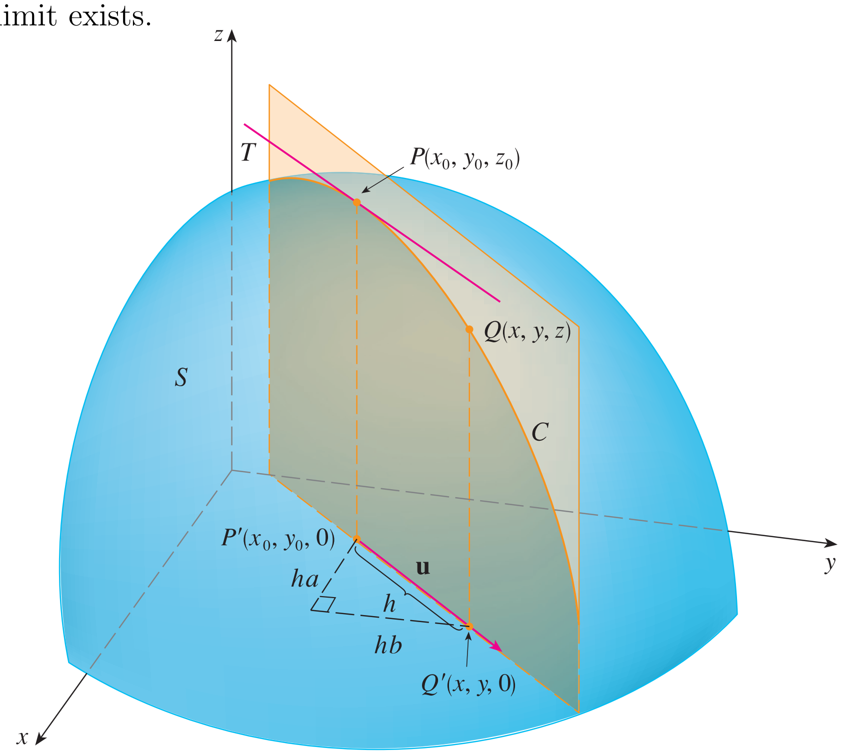
Example 9. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

14.6 Directional Derivatives and the Gradient

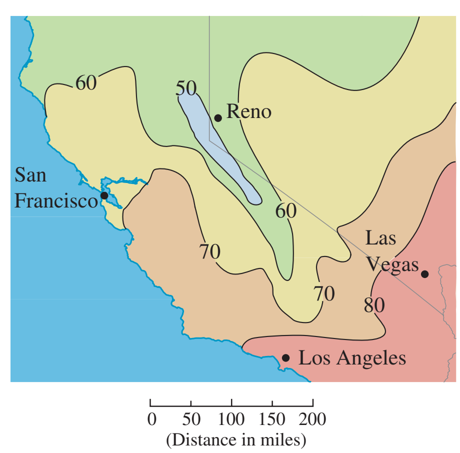
Definition 14.6.1. The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h},$$

if this limit exists.



Example 1. Use the weather map in the right figure to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.



Theorem 14.6.1. *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and*

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

Proof. If we define a function g of the single variable h by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of the derivative, we have

$$\begin{aligned} g'(0) &= \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \\ &= D_{\mathbf{u}} f(x_0, y_0). \end{aligned}$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0+ha$, $y = y_0+hb$, so the Chain Rule gives

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)a + f_y(x, y)b.$$

If we now put $h = 0$, then $x = x_0$, $y = y_0$, and

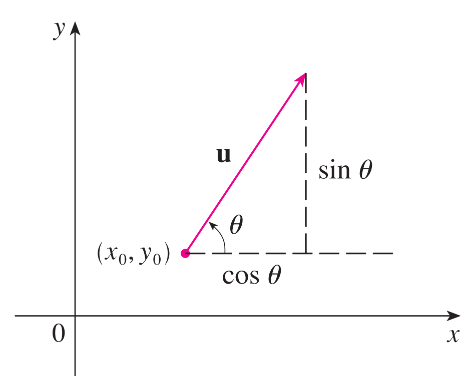
$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b.$$

Thus

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b.$$ □

Remark 1. If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis (as in the figure), then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta\rangle$ and the formula in Theorem 14.6.1 becomes

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$



Example 2. Find the directional derivative $D_{\mathbf{u}}f(x,y)$ if

$$f(x,y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}}f(1,2)$?

Definition 14.6.2. If f is a function of two variables x and y, then the gradient of f is the vector function ∇f (or $\mathbf{grad}f$) defined by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Example 3. If $f(x,y) = \sin x + e^{xy}$, then find $\nabla f(x,y)$ and $\nabla f(0,1)$.

Remark 2. With this notation for the gradient vector, we can rewrite the equation for the directional derivative of a differentiable function as

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u}.$$

This expresses the directional derivative in the direction of a unit vector $\mathbf{u}$ as the scalar projection of the gradient vector onto $\mathbf{u}$.

Example 4. Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

Definition 14.6.3. The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists. More compactly,

$$D_{\mathbf{u}}f(\mathbf{x}_0) = \lim_{h\to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where $\mathbf{x}_0 = \langle x_0, y_0\rangle$ if $n = 2$ and $\mathbf{x}_0 = \langle x_0, y_0, z_0\rangle$ if $n = 3$.

Remark 3. If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c\rangle$, then the same method that was used to prove Theorem 14.6.1 can be used to show that

$$D_{\mathbf{u}}f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c.$$

For a function of three variables, the gradient vector, denoted by ∇f or $\mathbf{grad} f$, is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle,$$

or, for short,

$$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Just as with functions of two variables, the directional derivative can be rewritten as

$$D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}.$$

Example 5. If $f(x, y, z) = x \sin yz$,

(a) find the gradient of f

(b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Theorem 14.6.2. *Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.*

Proof.

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos \theta = |\nabla f| \cos \theta$$

where θ is the angle between ∇f and $\mathbf{u}$. The maximum value of $\cos \theta$ is 1 and this occurs when $\theta = 0$. Therefore the maximum value of $D_{\mathbf{u}}f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when $\mathbf{u}$ has the same direction as ∇f. □

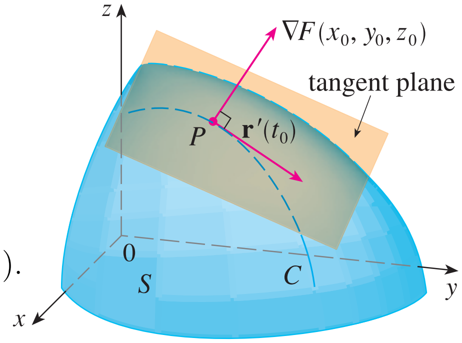
Example 6.

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q\left(\frac{1}{2}, 2\right)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Example 7. Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1+x^2+2y^2+3z^2)$, where T is measured in degrees Celsius and x, y, z, in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

Definition 14.6.4. If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, the tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ is the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. (See the figure.) Using the standard equation of a plane, we can write the equation of this tangent plane as



$$F_x(x_0, y_0, z_0)(x-x_0)+F_y(x_0, y_0, z_0)(y-y_0)+F_z(x_0, y_0, z_0)(x-x_0).$$

Definition 14.6.5. The normal line to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so its symmetric equations are

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}.$$

Example 8. Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

14.7 Maximum and Minimum Values

Definition 14.7.1. A function of two variables has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). The number $f(a, b)$ is called a local maximum value. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a local minimum at (a, b) and $f(a, b)$ is a local minimum value. If these inequalities hold for all points (x, y) in the domain of f, then f has an absolute maximum (or absolute minimum) at (a, b).

Theorem 14.7.1. *If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.*

Proof. Let $g(x) = f(x, b)$. If f has a local maximum (or minimum) at (a, b), then g has a local maximum (or minimum) at a, so $g'(a) = 0$ by Fermat's Theorem. But $g'(a) = f_x(a, b)$ and so $f_x(a, b) = 0$. Similarly, by applying Fermat's Theorem to the function $G(y) = f(a, y)$, we obtain $f_y(a, b) = 0$. □

Definition 14.7.2. A point (a, b) is called a critical point (or stationary point) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

Example 1. Find the extreme values of $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

Example 2. Find the extreme values of $f(x, y) = y^2 - x^2$.

Theorem 14.7.2 (Second Derivatives Test)**.** *Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let*

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

Remark 1. In case (c) the point (a, b) is called a saddle point of f and the graph of f crosses its tangent plane at (a, b).

Remark 2. If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

Remark 3. To remember the formula for D, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2.$$

Example 3. Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

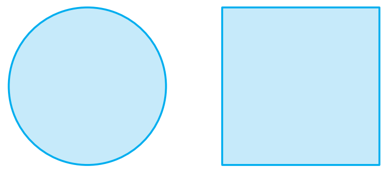
Example 4. Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4.$$

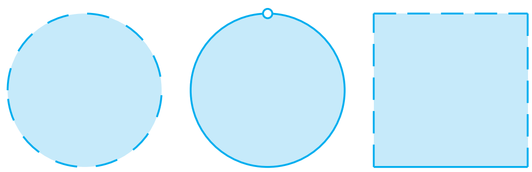
Example 5. Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

Example 6. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Definition 14.7.3. A closed set in $\mathbb{R}^2$ is one that contains all its boundary points. [A boundary point of D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D.]
A bounded set in $\mathbb{R}^2$ is one that is contained within some disk.



Closed sets



Sets that are not closed

Theorem 14.7.3 (Extreme Value Theorem for Functions of Two Variables). *If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.*

Remark 4. To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 7. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

14.8 Lagrange Multipliers

Theorem 14.8.1 (Method of Lagrange Multipliers)**.** *To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:*

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k.$$

The number λ is called a Lagrange multiplier.

(b) Evaluate f at the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

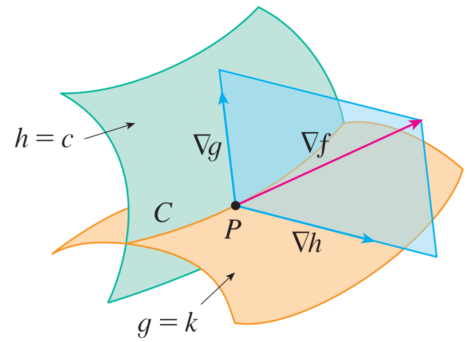
Example 1. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Example 2. Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Example 3. Find the extreme values of $f(x, y) = x^2+2y^2$ on the disk $x^2+y^2 \leq 1$.

Example 4. Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

Theorem 14.8.2 (Method of Lagrange Multipliers for Two Constraints)**.** *To find the maximum and minimum values of $f(x, y, z)$ subject to the constraints $g(x, y, z) = k$ and $h(x, y, z) = c$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$, $\nabla h \neq \mathbf{0}$, and ∇g is not parallel to ∇h]:*



(a) Find all values of x, y, z, λ, and μ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$

and

$$g(x, y, z) = k \qquad h(x, y, z) = c.$$

The numbers λ and μ are called Lagrange multipliers.

(b) Evaluate f at the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Example 5. Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Chapter 15

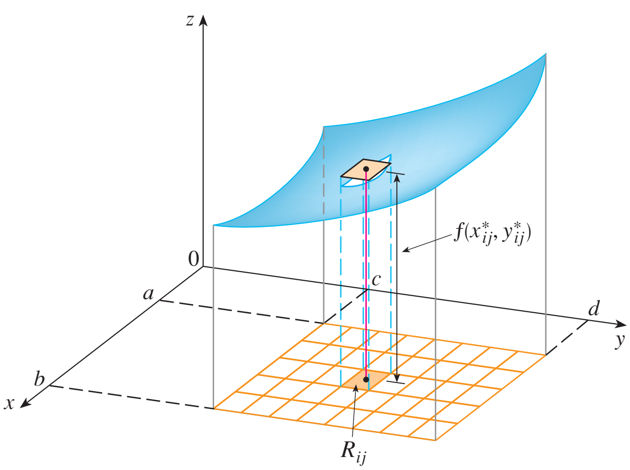
Multiple Integrals

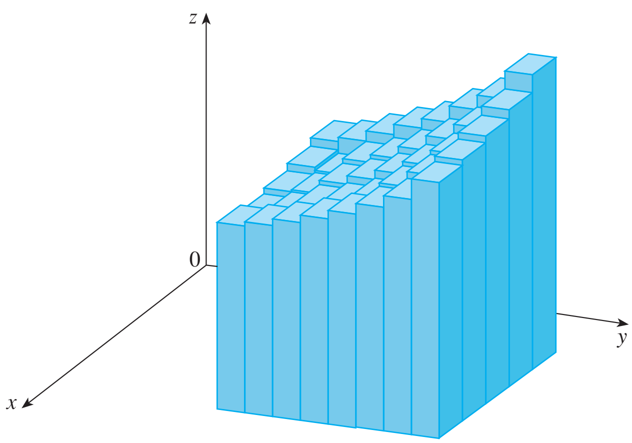
15.1 Double Integrals over Rectangles

Definition 15.1.1. The double integral of f over the rectangle R is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

if this limit exists. The points (x_{ij}^*, y_{ij}^*) are called sample points, $\Delta A = \Delta x\,\Delta y$ is the area of the subrectangle R_{ij} formed by the subintervals $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$, and the sum is called a double Riemann sum.





Definition 15.1.2. If $f(x, y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA.$$

Example 1. Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

Example 2. If $R = \{(x, y) \mid -1 \leq x \leq 1, -2 \leq y \leq 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA.$$

Theorem 15.1.1 (Midpoint Rule for Double Integrals)**.**

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_j)\,\Delta A$$

where $\bar{x}_i$ *is the midpoint of* $[x_{i-1},x_i]$ *and* $\bar{y}_i$ *is the midpoint of* $[y_{j-1},y_j]$.

Example 3. Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x-3y^2)\,dA$, where $R = \{(x,y) \mid 0 \le x \le 2, 1 \le y \le 2\}$.

Definition 15.1.3. Suppose that f is a function of two variables that is integrable on the rectangle $R = [a,b] \times [c,d]$. We use the notation $\int_a^b f(x,y)dx$ to mean that y is held fixed and $f(x,y)$ is integrated with respect to x from $x = a$ to $x = b$. This procedure is called partial integration with respect to x. Integrating this function gives us an <u>iterated integral</u>

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

where we first integrate with respect to x (holding y fixed) from $x = a$ to $x = b$ and then we integrate the resulting function of y with respect to y from $y = c$ to $y = d$.

Example 4. Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

(b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

Theorem 15.1.2 (Fubini's Theorem)**.** *If f is continuous on the rectangle $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$, then*

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy.$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 5. Evaluate the double integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \le x \le 2, 1 \le y \le 2\}$.

Example 6. Evaluate $\iint_R y \sin(xy)\, dA$, where $R = [1, 2] \times [0, \pi]$.

Example 7. Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

Theorem 15.1.3.

$$\iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy \qquad \textit{where } R = [a,b] \times [c,d].$$

Proof. By Fubini's Theorem,

$$\iint_R g(x)h(y)\,dA = \int_c^d \int_a^b g(x)h(y)\,dx\,dy = \int_c^d \left[\int_a^b g(x)h(y)\,dx\right] dy.$$

In the inner integral, y is a constant, so $h(y)$ is a constant and we can write

$$\int_c^d \left[\int_a^b g(x)h(y)\,dx\right] dy = \int_c^d \left[h(y)\left(\int_a^b g(x)\,dx\right)\right] dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

since $\int_a^b g(x)\,dx$ is a constant. □

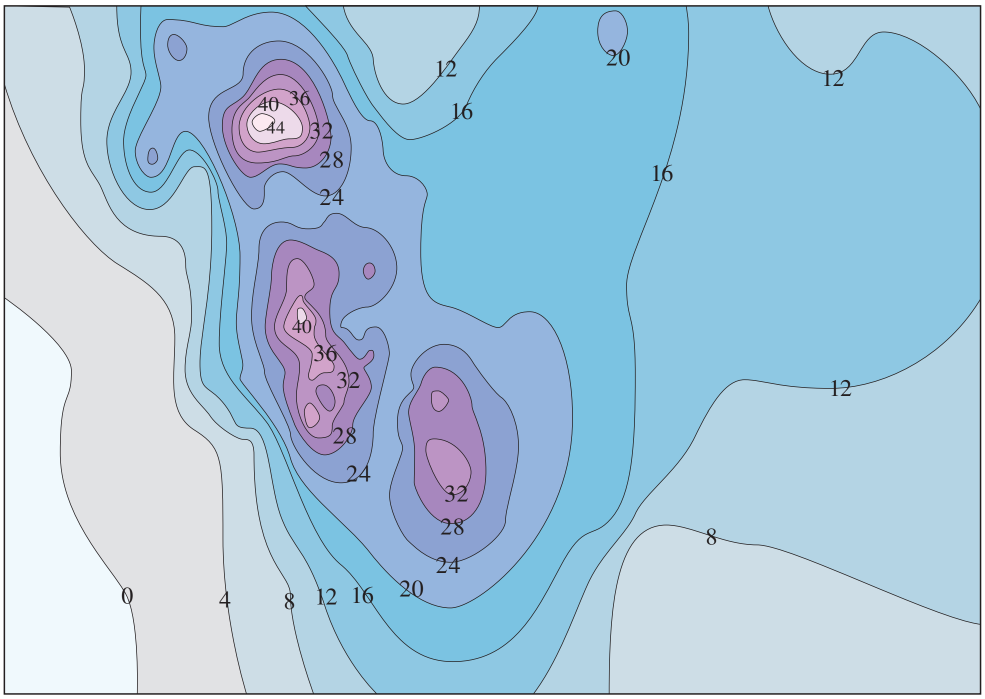
Example 8. Find $\iint_R \sin x \cos y\,dA$ if $R = [0, \pi/2] \times [0, \pi/2]$.

Definition 15.1.4. The average value of a function f of two variables defined on a rectangle R is

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)\,dA$$

where $A(R)$ is the area of R.

Example 9. The contour map in the figure shows the snowfall, in inches, that fell on the state of Colorado on December 20 and 21, 2006. (The state is in the shape of a rectangle that measures 388 mi west to east and 276 mi south to north.) Use the contour map to estimate the average snowfall for the entire state of Colorado on those days.



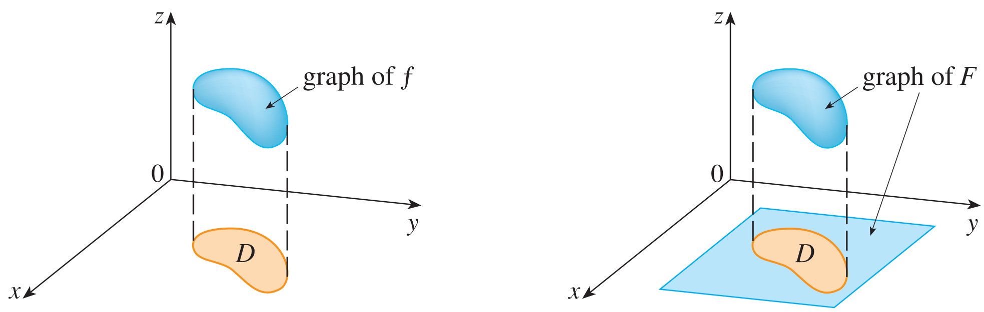
15.2 Double Integrals over General Regions

Definition 15.2.1. If F is integrable over R and D is a bounded region then we define the double integral of f over D by

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

where F is given by

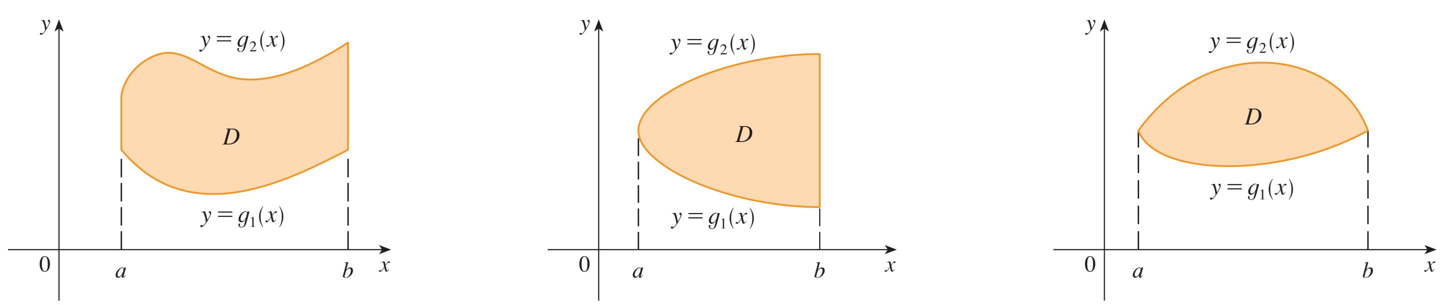
$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D, \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D. \end{cases}$$



Definition 15.2.2. A plane region D is said to be of type I if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous on $[a,b]$. Some examples of type I regions are shown in the figure.



Theorem 15.2.1. *If f is continuous on a type I region D such that*

$$D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

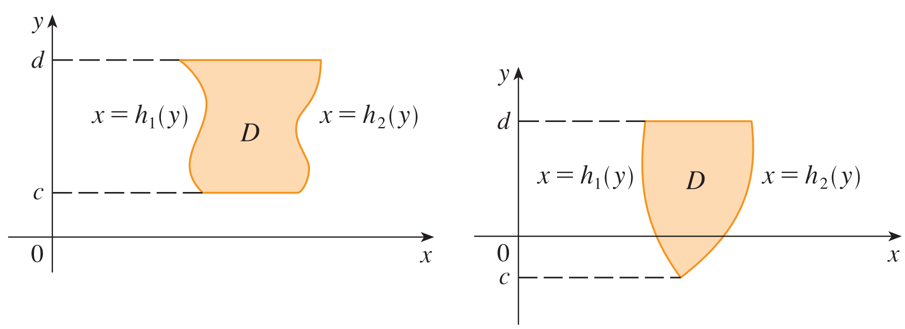
then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

Definition 15.2.3. A plane region D is said to be of type II if it lies between the graphs of two continuous functions of y, that is,

$$D = \{(x, y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous on $[c, d]$. Some examples of type II regions are shown in the figure.



Theorem 15.2.2. *If f is continuous on a type II region D such that*

$$D = \{(x, y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Example 1. Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Example 2. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Example 3. Evaluate $\iint_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Example 4. Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Example 5. Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

Theorem 15.2.3 (Properties of Double Integrals)**.**

1. $\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA.$

2. $\iint_D cf(x,y)\, dA = c \iint_D f(x,y)\, dA$ *where c is a constant.*

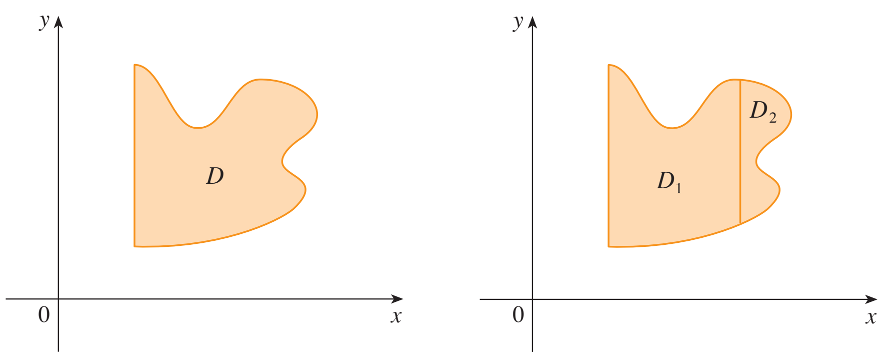
3. *If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then*

$$\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA.$$

4. *If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries, then*

$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$$

This property can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II, as illustrated by the figure.



(a) D is neither type I nor type II.
(b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

5. $\iint_D 1\, dA = A(D)$ *where $A(D)$ is the area of D.*

6. *If $m \leq f(x,y) \leq M$ for all (x,y) in D, then*

$$mA(D) \leq \iint_D f(x,y)\, dA \leq MA(D).$$

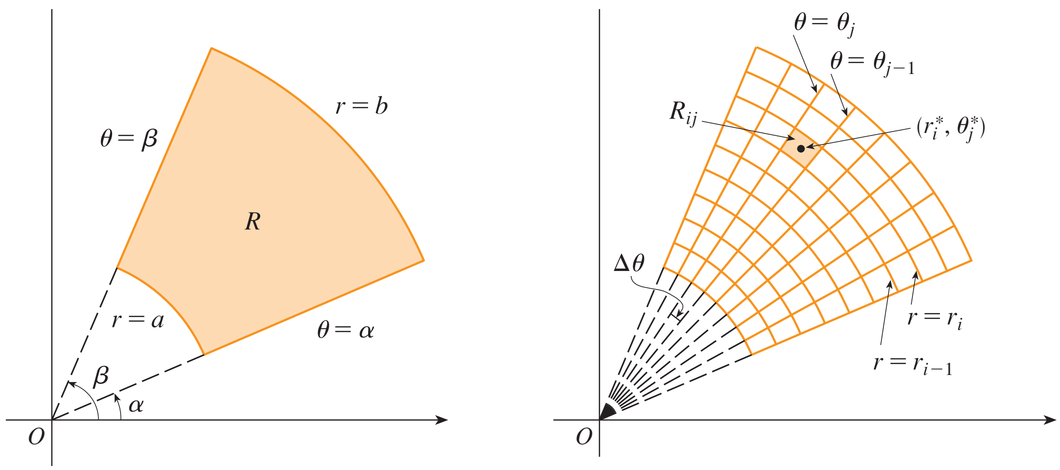
Example 6. Use Property 6 to estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and radius 2.

15.3 Double Integrals in Polar Coordinates

Definition 15.3.1. The region given by

$$R = \{(r, \theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$$

is called a polar rectangle, as shown in the figure.



Theorem 15.3.1 (Change to Polar Coordinates in a Double Integral)**.** *If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then*

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Proof. The "center" of the polar subrectangle

$$R_{ij} = \{(r, \theta) \mid r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\}$$

has polar coordinates

$$r_i^* = \frac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j).$$

Since the area of a sector of a circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$, the area of R_{ij} is

$$\begin{aligned}\Delta A_i &= \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta \\ &= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^*\Delta r\Delta\theta.\end{aligned}$$

Therefore we have

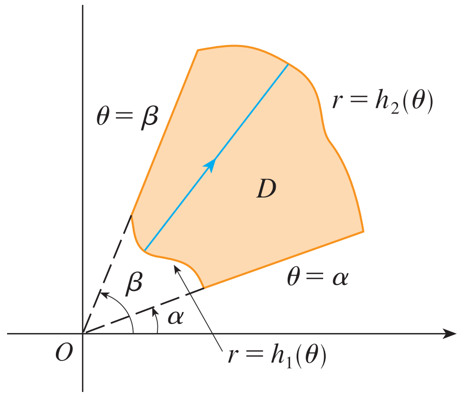
$$\begin{aligned}\iint_R f(x,y)\,dA &= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\,\Delta A_i \\ &= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta. \qquad \square\end{aligned}$$

Example 1. Evaluate $\iint_R (3x+4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Example 2. Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z=1-x^2-y^2$.

Theorem 15.3.2. *If f is continuous on a polar region of the form*

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$



then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Example 3. Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Example 4. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

15.4 Applications of Double Integrals

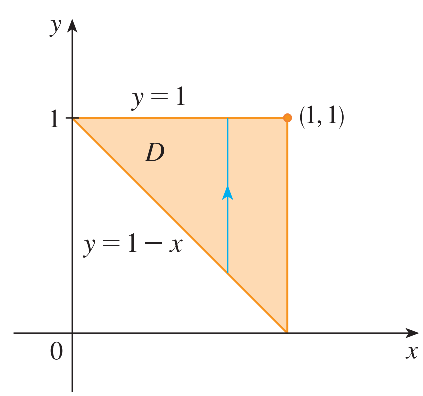
Definition 15.4.1. Suppose a lamina occupies a region D of the xy-plane and its density (in units of mass per unit area) at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D. Then the total mass of the lamina is given by

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D \rho(x, y)\, dA.$$

Similarly, if an electric charge is distributed over a region D and the charge density (in units of charge per unit area) is given by $\sigma(x, y)$ at a point (x, y) in D, then the total charge Q is given by

$$Q = \iint_D \sigma(x, y)\, dA.$$

Example 1. Charge is distributed over the triangular region D in the figure so that the charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.



Definition 15.4.2. Suppose a lamina occupies a region D and has density function $\rho(x, y)$. The moment of the lamina about the x-axis is

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^{*} \rho(x_{ij}^{*}, y_{ij}^{*})\, \Delta A = \iint_D y\rho(x, y)\, dA.$$

Similarly, moment about the y-axis is

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^{*} \rho(x_{ij}^{*}, y_{ij}^{*})\, \Delta A = \iint_D x\rho(x, y)\, dA.$$

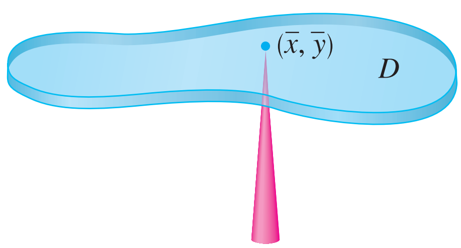
Definition 15.4.3. The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\rho(x, y)\, dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y\rho(x, y)\, dA$$

where the mass m is given by

$$m = \iint_D \rho(x, y)\, dA.$$

The lamina balances horizontally when supported at its center of mass (see the figure).



Example 2. Find the mass and center of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$, and $(0,2)$ if the density function is $\rho(x,y) = 1 + 3x + y$.

Example 3. The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

Definition 15.4.4. The moment of inertia (also called the second moment) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. The moment of inertia of the lamina about the x-axis is defined to be

$$I_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D y^2 \rho(x,y)\, dA.$$

Similarly, the moment of inertia about the y-axis is defined to be

$$I_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D x^2 \rho(x,y)\, dA.$$

The moment of inertia about the origin, also called the polar moment of inertia is defined to be

$$I_0 = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D (x^2+y^2) \rho(x,y)\, dA.$$

Example 4. Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x,y) = \rho$, center the origin, and radius a.

Definition 15.4.5. The radius of gyration of a lamina about an axis is the number R such that

$$mR^2 = I$$

where m is the mass of the lamina and I is the moment of inertia about the given axis. In particular, the radius of gyration $\bar{\bar{y}}$ with respect to the x-axis and the radius of gyration $\bar{\bar{x}}$ with respect to the y-axis are given by the equations

$$m\bar{\bar{y}}^2 = I_x \qquad m\bar{\bar{x}}^2 = I_y.$$

Example 5. Find the radius of gyration about the x-axis of the disk in Example 4.

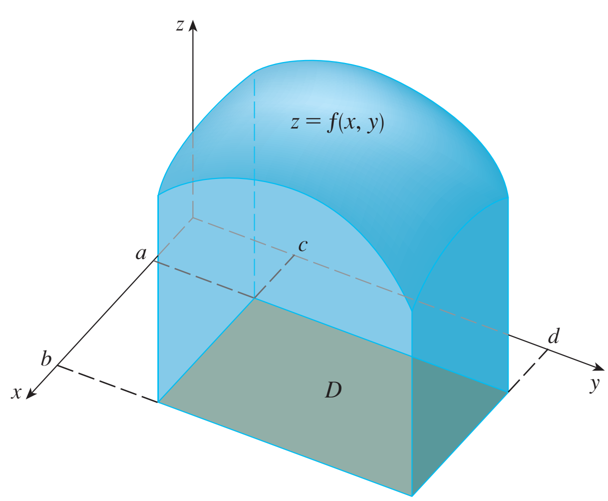
Definition 15.4.6. The joint density function of two continuous random variables X and Y is a function f of two variables such that the probability that (X, Y) lies in a region D is

$$P((X,Y) \in D) = \iint_D f(x,y)\,dA.$$

In particular, if the region is a rectangle, the probability that X lies between a and b and Y lies between c and d is

$$P(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

(See the figure.)



Remark 1. Because probabilities aren't negative and are measured on a scale from 0 to 1, the joint density function has the following properties:

$$f(x,y) \geq 0 \qquad \iint_{\mathbb{R}^2} f(x,y)\,dA = 1$$

for

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \lim_{a\to\infty}\iint_{D_a} f(x,y)\,dA$$

where D_a is the disk with radius a and center the origin.

Example 6. If the joint density function for X and Y is given by

$$f(x,y) = \begin{cases} C(x+2y) & \text{if } 0 \leq x \leq 10,\ 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

find the value of the constant C. Then find $P(X \leq 7, Y \geq 2)$.

Definition 15.4.7. Suppose X is a random variable with probability density function $f_1(x)$ and Y is a random variable with density function $f_2(y)$. Then X and Y are called independent random variables if their joint density function is the product of their individual density functions:

$$f(x,y) = f_1(x)f_2(y).$$

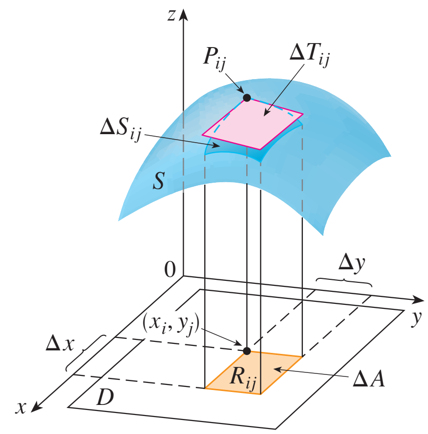
Example 7. The manager of a movie theater determines that the average time moviegoers wait in line to buy a ticket for this week's film is 10 minutes and the average time they wait to buy popcorn is 5 minutes. Assuming that the waiting times are independent, find the probability that a moviegoer waits a total of less than 20 minutes before taking his or her seat.

Definition 15.4.8. If X and Y are random variables with joint density function f, we define the X-mean and Y-mean, also called the expected values of X and Y, to be

$$\mu_1 = \iint_{\mathbb{R}^2} xf(x,y)\,dA \qquad \mu_2 = \iint_{\mathbb{R}^2} yf(x,y)\,dA.$$

Example 8. A factory produces (cylindrically shaped) roller bearings that are sold as having diameter 4.0 cm and length 6.0 cm. In fact, the diameters X are normally distributed with mean 4.0 cm and standard deviation 0.01 cm while the lengths Y are normally distributed with mean 6.0 cm and standard deviation 0.01 cm. Assuming that X and Y are independent, write the joint density function and graph it. Find the probability that a bearing randomly chosen from the production line has either length or diameter that differs from the mean by more than 0.02 cm.

15.5 Surface Area



Definition 15.5.1. Let S be a surface with equation $z = f(x, y)$, where f has continuous partial derivatives. We define the surface area of S to be

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

where ΔT_{ij} is the part of the tangent plane to S at the point P_{ij} on the surface corresponding to a rectangle R_{ij} in the domain D of f.

Theorem 15.5.1. *The area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are continuous, is*

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA.$$

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be the vectors that start at P_{ij} and lie along the sides of the parallelogram with area ΔT_{ij}. Then $\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|$. Since $f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ are the slopes of the tangent lines through P_{ij} in the directions of $\mathbf{a}$ and $\mathbf{b}$, we have

$$\begin{aligned} \mathbf{a} &= \Delta x\, \mathbf{i} + f_x(x_i, y_j)\, \Delta x\, \mathbf{k} \\ \mathbf{b} &= \Delta y\, \mathbf{j} + f_y(x_i, y_j)\, \Delta y\, \mathbf{k}. \end{aligned}$$

and

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x(x_i, y_j)\, \Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\, \Delta y \end{vmatrix} \\ &= -f_x(x_i, y_j)\, \Delta x\, \Delta y\, \mathbf{i} - f_y(x_i, y_j)\, \Delta x\, \Delta y\, \mathbf{j} + \Delta x\, \Delta y\, \mathbf{k} \\ &= [-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}]\, \Delta A. \end{aligned}$$

Thus

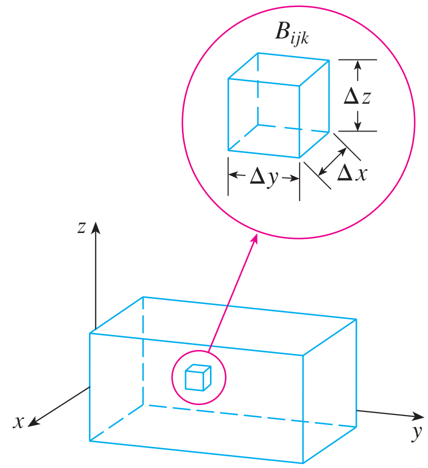
$$\begin{aligned} A(S) &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij} = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{a} \times \mathbf{b}| \\ &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\, \Delta A. \end{aligned}$$

□

Example 1. Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

Example 2. Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

15.6 Triple Integrals



Definition 15.6.1. The triple integral of f over the box B is

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta V$$

if this limit exists. The points $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ are called sample points, $\Delta V = \Delta x\,\Delta y\,\Delta z$ is the volume of the sub-box $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$, and the sum is called a triple Riemann sum.

Theorem 15.6.1 (Fubini's Theorem for Triple Integrals)**.** *If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$, then*

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

Example 1. Evaluate the triple integral $\iiint_R xyz^2\,dV$ where B is the rectangular box given by

$$B = \{(x,y,z) \mid 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}.$$

Definition 15.6.2. If F is integrable over B and E is a bounded region then we define the triple integral of f over E by

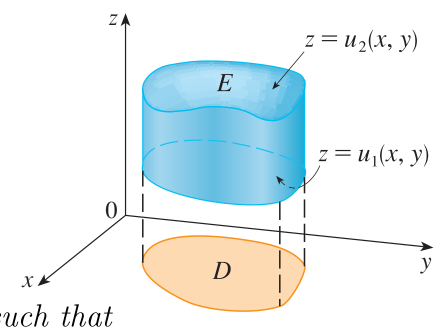
$$\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV$$

where F is defined so that it agrees with f on E but is 0 for points in B that are outside E.

Definition 15.6.3. A solid region E is said to be of type 1 if it lies between the graphs of two continuous functions of x and y, that is

$$E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \leq z \leq u_2(x,y)\}$$

where D is the projection of E onto the xy-plane as shown in the figure.



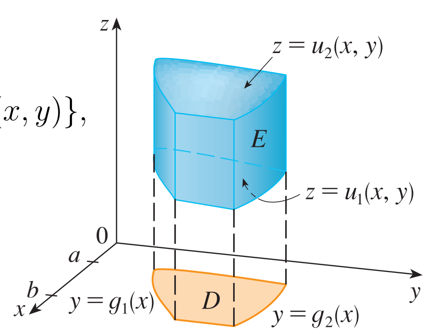
Theorem 15.6.2. *If f is continuous on a type 1 region E such that*

$$E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \leq z \leq u_2(x,y)\}$$

then

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA.$$

Remark 1. If the projection D of E onto the xy-plane is a type I plane region (as in the figure), then

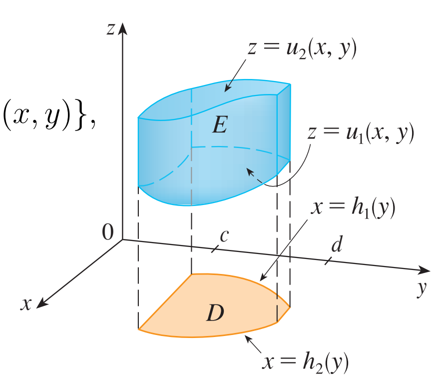


$$E = \{(x,y,z) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x,y) \leq z \leq u_2(x,y)\},$$

so

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

If, on the other hand, D is a type II plane region (as in the figure), then



$$E = \{(x,y,z) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y), u_1(x,y) \leq z \leq u_2(x,y)\},$$

so

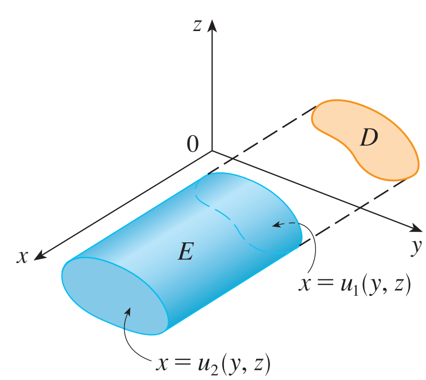
$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy.$$

Example 2. Evaluate $\iiint_E z\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Definition 15.6.4. A solid region E is of type 2 if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

where D is the projection of E onto the yz-plane as shown in the figure.

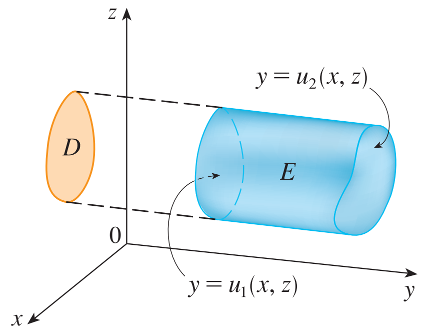


Theorem 15.6.3. *If f is continuous on a type 2 region E such that*

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA.$$



Definition 15.6.5. A solid region E is of type 3 if it is of the form

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

where D is the projection of E onto the xz-plane as shown in the figure.

Theorem 15.6.4. *If f is continuous on a type 3 region E such that*

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA.$$

Example 3. Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Example 4. Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x, then z, and then y.

Theorem 15.6.5.

$$V(E) = \iiint_E dV.$$

Example 5. Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Definition 15.6.6. If the density function of a solid object that occupies the region E is $\rho(x, y, z)$, in units of mass per unit volume, at any given point (x, y, z), then its mass is

$$m = \iiint_E \rho(x, y, z)\, dV$$

and its moments about the three coordinate planes are

$$M_{yz} = \iiint_E x\rho(x, y, z)\, dV \qquad M_{xz} = \iiint_E y\rho(x, y, z)\, dV$$
$$M_{xy} = \iiint_E z\rho(x, y, z)\, dV.$$

The center of mass is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}.$$

If the density is constant, the center of mass of the solid is called the centroid of E. The moments of inertia about the three coordinate axes are

$$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\, dV \qquad I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)\, dV$$
$$I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\, dV.$$

Definition 15.6.7. The total electric charge on a solid object occupying a region E and having charge density $\sigma(x, y, z)$ is

$$Q = \iiint_E \sigma(x, y, z)\, dV.$$

Definition 15.6.8. If we have three continuous random variables X, Y, and Z, their joint density function is a function of three variables such that the probability that (X, Y, Z) lies in E is

$$P((X, Y, Z) \in E) = \iiint_E f(x, y, z)\, dV.$$

In particular,

$$P(a \leq X \leq b, c \leq Y \leq d, r \leq Z \leq s) = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx.$$

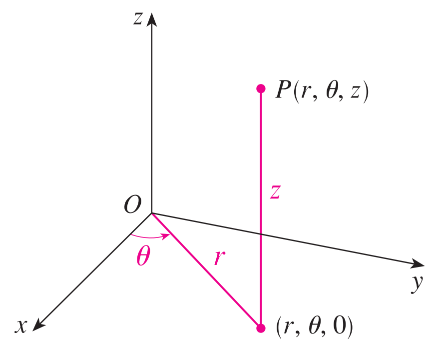
The joint density function satisfies

$$f(x, y, z) \geq 0 \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\, dz\, dy\, dx = 1.$$

Example 6. Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.

15.7 Integrals in Cylindrical Coordinates

Definition 15.7.1. In the cylindrical coordinate system, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P. (See the figure.)



Theorem 15.7.1. *To convert from cylindrical to rectangular coordinates, we use the equations*

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

whereas to convert from rectangular to cylindrical coordinates, we use

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z.$$

Example 1. (a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.

(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

Example 2. Describe the surface whose equation in cylindrical coordinates is $z = r$.

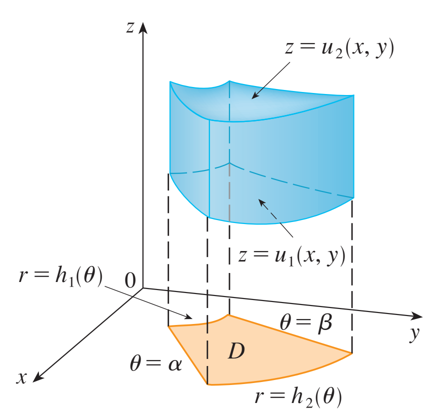
Theorem 15.7.2. *Suppose that E is a type 1 region whose projection D onto the xy-plane is described in polar coordinates (see the figure). In particular, suppose that f is continuous and*

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is given in polar coordinates by

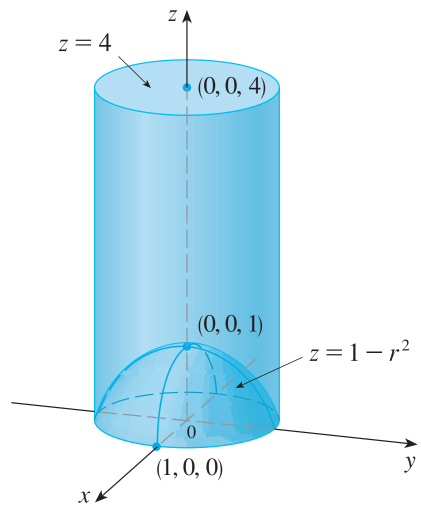
$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}.$$

Then the formula for triple integration in cylindrical coordinates is



$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta.$$

Example 3. A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See the figure.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

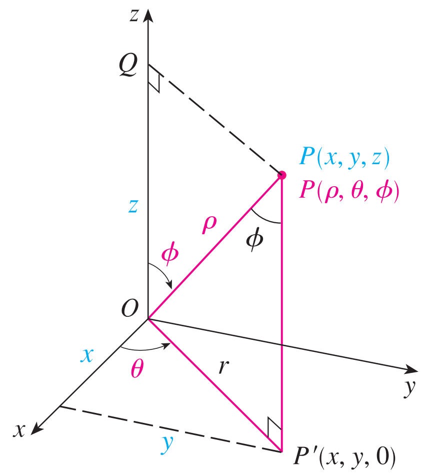


Example 4. Evaluate $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx$.

15.8 Integrals in Spherical Coordinates

Definition 15.8.1. The spherical coordinates (ρ, θ, ϕ) of a point P in space are shown in the figure, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP. Note that

$$\rho \geq 0 \qquad 0 \leq \phi \leq \pi.$$



Theorem 15.8.1. *The relationship between rectangular and spherical coordinates can be seen from the figure. To convert from spherical to rectangular coordinates, we use the equations*

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi.$$

To convert from rectangular to spherical coordinates, we use the equation

$$\rho^2 = x^2 + y^2 + z^2.$$

Example 1. The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

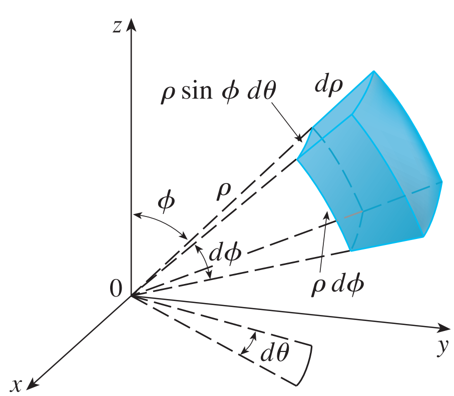
Example 2. The point $\left(0, 2\sqrt{3}, -2\right)$ is given in rectangular coordinates. Find spherical coordinates for this point.

Theorem 15.8.2. *The formula for triple integration in spherical coordinates is*

$$\iiint_E f(x,y,z)\,dV$$
$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

where E is a spherical wedge given by

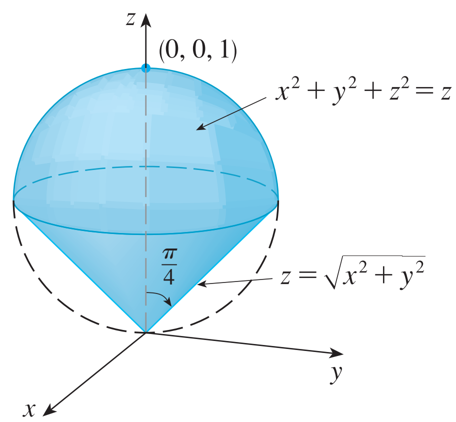
$$E = \{(\rho,\theta,\phi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}.$$



Example 3. Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}.$$

Example 4. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. (See the figure.)



15.9 Change of Variables in Multiple Integrals

Definition 15.9.1. A change of variables is given by a transformation T from the uv-plane to the xy-plane:

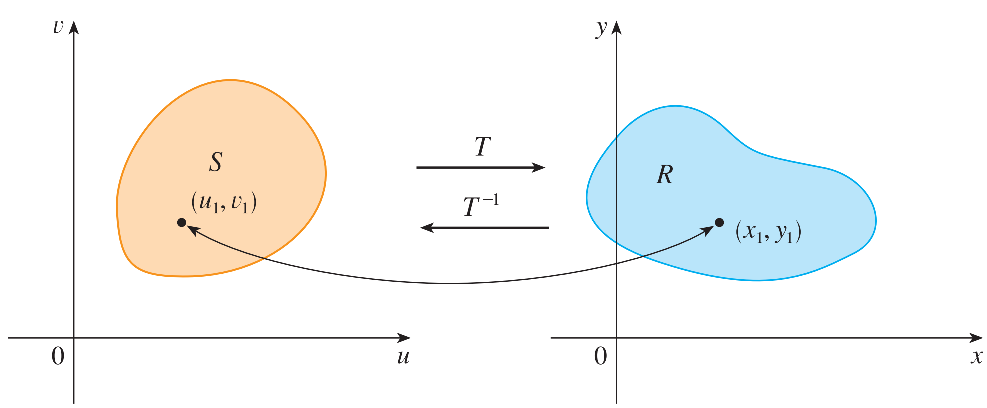
$$T(u, v) = (x, y)$$

where x and y are related to u and v by the equations

$$x = g(u, v) \qquad y = h(u, v).$$

We usually assume that T is a C^1 transformation, which means that g and h have continuous first-order partial derivatives.

Remark 1. A transformation T is really just a function whose domain and range are both subsets of $\mathbb{R}^2$. If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the image of the point (u_1, v_1). If no two points have the same image, T is called one-to-one. The figure shows the effect of a transformation T on a region S in the uv-plane. T transforms S into a region R in the xy-plane called the image of S, consisting of the images of all points in S.



If T is a one-to-one transformation, then it has an inverse transformation T^{-1} from the xy-plane to the uv-plane and it may be possible to solve for u and v in terms of x and y:

$$u = G(x, y) \qquad v = H(x, y).$$

Example 1. A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv.$$

Find the image of the square $S = \{(u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq 1\}$.

Definition 15.9.2. The Jacobian of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Remark 2. This notation can be used to show that the area ΔA of the image R in the xy-plane of a rectangle in the uv-plane is approximately

$$\Delta A \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \, \Delta v.$$

Theorem 15.9.1 (Change of Variables in a Double Integral)**.** *Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then*

$$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv.$$

Example 2. Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\, dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.

Example 3. Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\, dA$ where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$.

Definition 15.9.3. The Jacobian of the transformation T given by $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$ is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Theorem 15.9.2 (Change of Variables in a Triple Integral)**.** *Under hypotheses similar to those in Theorem 15.9.1,*

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw.$$

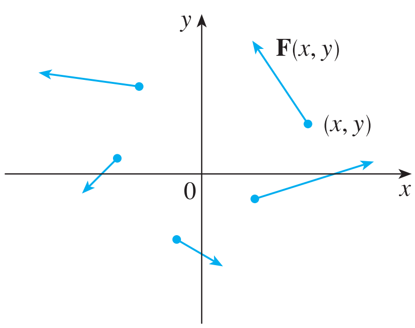
Example 4. Use Theorem 15.9.2 to derive the formula for triple integration in spherical coordinates.

Chapter 16

Vector Calculus

16.1 Vector Fields

Definition 16.1.1. Let D be a set in $\mathbb{R}^2$ (a plane region). A vector field on $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.



Remark 1. Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its component functions P and Q as follows:

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

or, for short,

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j}.$$

Note that P and Q are scalar functions of two variables and are sometimes called scalar fields to distinguish them from vector fields.

Definition 16.1.2. Let E be a subset of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

Remark 2. We can express a vector field $\mathbf{F}$ on $\mathbb{R}^3$ in terms of its component functions P, Q, and R as

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

Example 1. A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$. Describe $\mathbf{F}$ by sketching some of the vectors $\mathbf{F}(x, y)$.

Example 2. Sketch the vector field on $\mathbb{R}^3$ given by $\mathbf{F}(x, y, z) = z\mathbf{k}$.

Example 3. Imagine a fluid flowing steadily along a pipe and let $\mathbf{V}(x, y, z)$ be the velocity vector at a point (x, y, z). Then $\mathbf{V}$ assigns a vector to each point (x, y, z) in a certain domain E (the interior of the pipe) and so $\mathbf{V}$ is a vector field on $\mathbb{R}^3$ called a velocity field. Sketch a possible velocity field in a fluid flow.

Example 4. Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses m and M is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

where r is the distance between the objects and G is the gravitational constant. Let's assume that the object with mass M is located at the origin in $\mathbb{R}^3$ and let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z \rangle$. Write and sketch an equation for the gravitational field $\mathbf{F}$.

Example 5. Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the magnitude of the electric force $\mathbf{F}(\mathbf{x})$ exerted by this charge on a charge q located at a point (x, y, z) with position vector $\mathbf{x} = \langle x, y, z \rangle$ is

$$|\mathbf{F}| = \frac{\varepsilon q Q}{r^2}$$

where ε is a constant (that depends on the units used). This vector field and the one in Example 4 are examples of force fields. Instead of considering the electric force $\mathbf{F}$, physicists often consider the force per unit charge $\mathbf{E}(\mathbf{x}) = \frac{1}{q}\mathbf{F}(\mathbf{x})$, called the electric field of Q. Write equations for $\mathbf{F}$ and $\mathbf{E}$.

Definition 16.1.3. If f is a scalar function of two variables, its gradient

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

is a vector field on $\mathbb{R}^2$ called a gradient vector field. Likewise, if f is a scalar function of two variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Example 6. Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

Definition 16.1.4. A vector field $\mathbf{F}$ is called a <u>conservative vector field</u> if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a <u>potential function</u> for $\mathbf{F}$.

16.2 Line Integrals

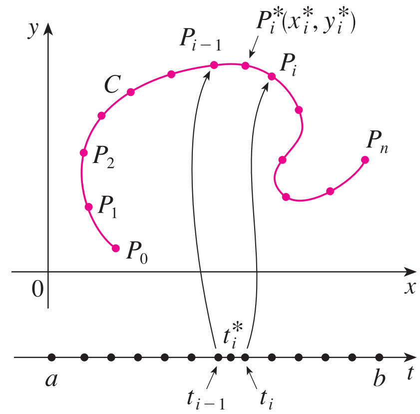
Definition 16.2.1. If f is defined on a smooth curve C given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad a \leq t \leq b,$$

then the line integral of f along C is

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

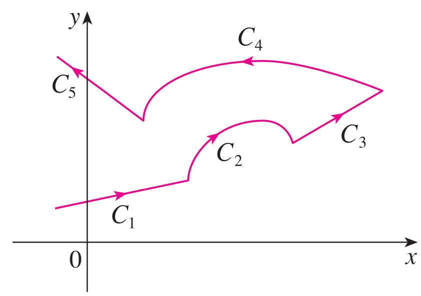
if this limit exists. The lengths Δs_i are of subarcs of C and the points (x_i^*, y_i^*) are sample points in the ith subarc.



Remark 1. Using the formula for the length of C we can write

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example 1. Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.



Definition 16.2.2. Suppose that C is a piecewise-smooth curve; that is, C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where, as illustrated in the figure, the initial point of C_{i+1} is the terminal point of C_i. Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x, y)\, ds = \int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds + \cdots + \int_{C_n} f(x, y)\, ds.$$

Example 2. Evaluate $\int_C 2x\, ds$ where C consists of the arc C_1 of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the vertical line segment C_2 from $(1, 1)$ to $(1, 2)$.

Definition 16.2.3. Suppose that $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve C. Then the mass m of the wire is given by

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*)\, \Delta s_i = \int_C \rho(x, y)\, ds.$$

The center of mass of the wire with density function ρ is located at the point $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{m} \int_C x\rho(x, y)\, ds \qquad \bar{y} = \frac{1}{m} \int_C y\rho(x, y)\, ds.$$

Example 3. A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geq 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

Definition 16.2.4. The integrals

$$\int_C f(x,y)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta x_i$$
$$\int_C f(x,y)\,dy = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta y_i$$

are called the line integrals of f along C with respect to x and y. The original line integral $\int_C f(x,y)\,ds$ is called the line integral with respect to arc length.

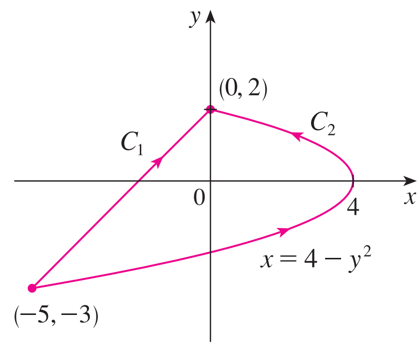
Theorem 16.2.1. *Line integrals with respect to x and y can also be evaluated by expressing everything in terms of t:*

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))x'(t)\,dt$$
$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))y'(t)\,dt.$$

Remark 2. When line integrals with respect to x and y occur together we abbreviate by writing

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)dx + Q(x,y)\,dy.$$

Example 4. Evaluate $\int_C y^2\,dx + x\,dy$, where (See the figure.)



(a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$.

Definition 16.2.5. Suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b,$$

or by a vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. If f is a function three variables that is continuous on some region containing C, then the line integral of f along C (with respect to arc length) is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta s_i$$

if this limit exists.

Remark 3. Using the formula for the length of C we can write

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt,$$

or, more compactly,

$$\int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt.$$

For the special case $f(x,y,z) = 1$, we get

$$\int_C ds = \int_a^b |\mathbf{r}'(t)|\,dt = L$$

where L is the length of the curve C.

Definition 16.2.6. The integrals

$$\int_C f(x,y,z)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta x_i = \int_a^b f(x(t),y(t),z(t))x'(t)\,dt$$
$$\int_C f(x,y,z)\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta y_i = \int_a^b f(x(t),y(t),z(t))y'(t)\,dt$$
$$\int_C f(x,y,z)\,dz = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta z_i = \int_a^b f(x(t),y(t),z(t))z'(t)\,dt$$

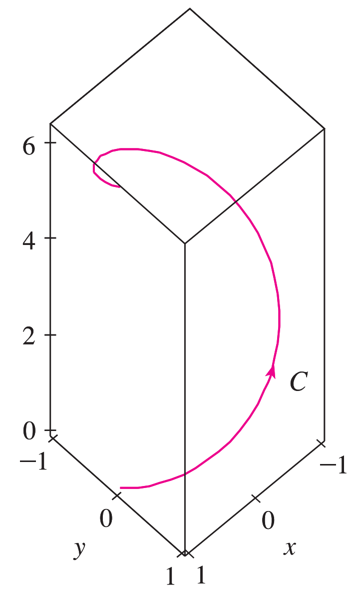
are called the line integrals of f along C with respect to x, y, and z.

Remark 4. As with line integrals in the plane, we evaluate integrals of the form

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

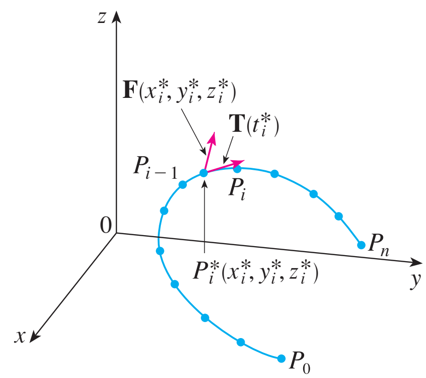
by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t.

Example 5. Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq 2\pi$. (See the figure.)



Example 6. Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C consists of the line segment C_1 from $(2, 0, 0)$ to $(3, 4, 5)$, followed by the vertical line segment C_2 from $(3, 4, 5)$ to $(3, 4, 0)$.

Definition 16.2.7. Suppose that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a continuous force field on $\mathbb{R}^3$. We define the work W done by the force field $\mathbf{F}$ as the limit of the Riemann sums



$$\sum_{i=1}^{n} [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)] \, \Delta s_i$$

where $P_i^*(x_i^*, y_i^*, z_i^*)$ is a point on the ith subarc $P_{i-1}P_i$ of C, and $\mathbf{T}(x, y, z)$ is the unit tangent vector at the point (x, y, z) on C. That is,

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) \, ds = \int_C \mathbf{F} \cdot \mathbf{T} \, ds.$$

Remark 5. If the curve C is given by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$, so

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)| \, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt,$$

which we abbreviate as $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Definition 16.2.8. Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} \, ds.$$

Example 7. Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \leq t \leq \pi/2$.

Example 8. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leq t \leq 1.$$

Theorem 16.2.2. *Suppose the vector field* $\mathbf{F}$ *on* $\mathbb{R}^3$ *is given in component form by* $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$*. Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz.$$

Proof.

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\ &= \int_a^b (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k})\,dt \\ &= \int_a^b \left[P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)\right] dt\end{aligned}$$

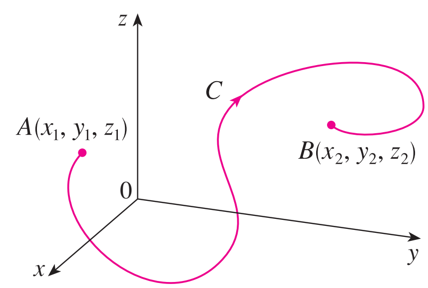
□

16.3 Fundamental Theorem for Line Integrals

Theorem 16.3.1 (Fundamental Theorem for Line Integrals)**.** *Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then*

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Proof. If f is a function of three variables and C is a space curve joining the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$, as in the figure, then the theorem becomes



$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1).$$

In this case (the case for two variables is similar),

$$\begin{aligned}\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt \\ &= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\,dt \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a)).\end{aligned}$$

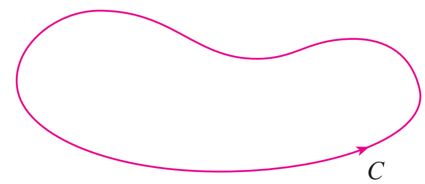
□

Example 1. Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

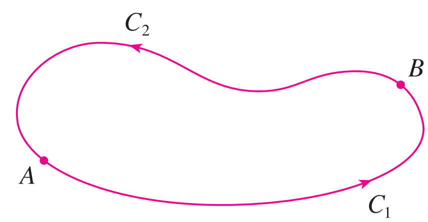
in moving a particle with mass m from the point $(3, 4, 12)$ to the point $(2, 2, 0)$ along a piecewise-smooth curve C. (See Example 16.1.4.)

Remark 1. In general, if $\mathbf{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is <u>independent of path</u> if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D that have the same initial points and the same terminal points. By Theorem 16.3.1, line integrals of conservative vector fields are independent of path. A curve is called <u>closed</u> if its terminal point coincides with its initial point, that is, $\mathbf{r}(b) = \mathbf{r}(a)$. (See the figure.)



Theorem 16.3.2. *$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.*

Proof. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D and C is any closed path in D, we can choose any two points A and B on C as being composed of the path C_1 from A to B followed by the path C_2 from B to A. (See the figure.) Then



$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

since C_1 and $-C_2$ have the same initial and terminal points.
Conversely, if it is true that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ whenever C is a closed path in D, then we demonstrate independence of path as follows. Take any two paths C_1 and C_2 from A to B in D and define C to be the curve consisting of C_1 followed by $-C_2$. Then

$$0 = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

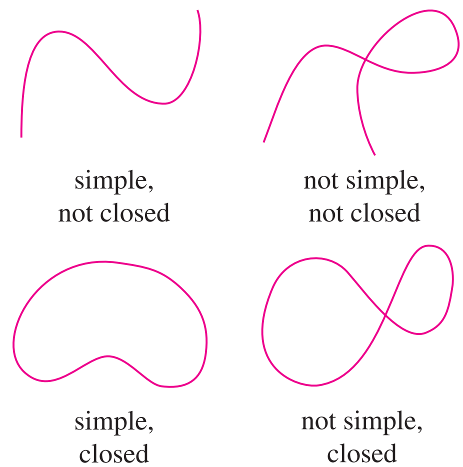
and so $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. □

Theorem 16.3.3. *Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. (By <u>open</u> we mean that for every point P in D there is a disk with center P that lies entirely in D, and by <u>connected</u> we mean that any two points in D can be joined by a path that lies in D.) If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.*

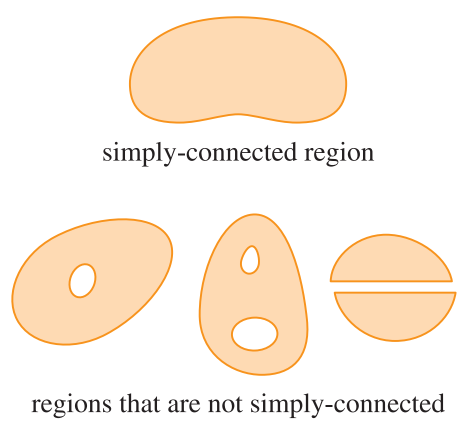
Theorem 16.3.4. *If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Definition 16.3.1. A simple curve is a curve that does not intersect itself anywhere between its endpoints. [See the figure; $\mathbf{r}(a) = \mathbf{r}(b)$ for a simple closed curve, but $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_2 < t_2 < b$.]



Definition 16.3.2. A simply-connected region in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D. [See the figure; a simply-connected region contains no hole and cannot consist of two separate pieces.]



Theorem 16.3.5. *Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D.$$

Then $\mathbf{F}$ is conservative.

Example 2. Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x - 2)\mathbf{j}$$

is conservative.

Example 3. Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$$

is conservative.

Example 4. (a) If $\mathbf{F}(x,y) = (3+2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

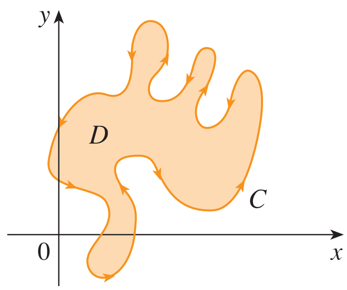
(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$$\mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j} \qquad 0 \le t \le \pi.$$

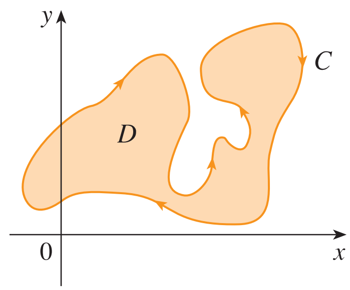
Example 5. If $\mathbf{F}(x, y, z) = y^2\mathbf{i} + (2xy + e^{3z})\mathbf{j} + 3ye^{3z}\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

16.4 Green's Theorem

Definition 16.4.1. The positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C. Thus if C is given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C. (See the figure.)



(a) Positive orientation



(b) Negative orientation

Theorem 16.4.1 (Green's Theorem)**.** *Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then*

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Remark 1. The notation

$$\oint P\,dx + Q\,dy \qquad \text{or} \qquad \circlearrowleft\!\!\!\!\int P\,dx + Q\,dy$$

is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C. Another notation for the positively oriented boundary curve of D is ∂D, so the equation in Green's Theorem can be written as

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P\,dx + Q\,dy.$$

Example 1. Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.

Example 2. Evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$, where C is the circle $x^2 + y^2 = 9$.

Theorem 16.4.2. *The area of a region D is*

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

Proof. Since the area of D is $\iint_D 1\,dA$, we wish to choose P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

There are several possibilities:

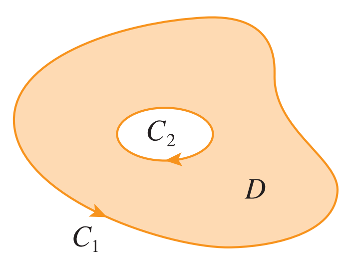
$$P(x,y) = 0 \qquad P(x,y) = -y \qquad P(x,y) = -\frac{1}{2}y$$
$$Q(x,y) = x \qquad Q(x,y) = 0 \qquad Q(x,y) = \frac{1}{2}x.$$

Then the result follows by Green's Theorem. □

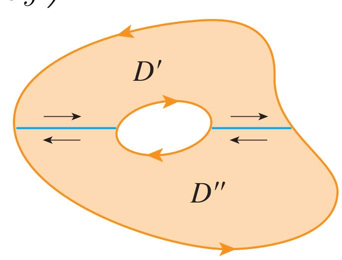
Example 3. Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Example 4. Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Remark 2. Green's Theorem can be extended to apply to regions with holes, that is, regions that are not simply-connected. Observe that the boundary C of the region D in the top figure consists of two simple closed curves C_1 and C_2. By dividing the region D into two regions D' and D'' by means of the lines shown in the bottom figure, and then applying Green's Theorem to each of D' and D'', we get



$$\begin{aligned}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\ &= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy \\ &= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy \\ &= \int_C P\,dx + Q\,dy.\end{aligned}$$



Example 5. If $\mathbf{F}(x, y) = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

16.5 Curl and Divergence

Definition 16.5.1. If $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

Remark 1. The equation for curl can be rewritten using operator notation by introducing the vector differential operator ∇ ("del") as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

It has meaning when it operates on a scalar function to produce the gradient of f:

$$\nabla f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field $\mathbf{F}$ as follows:

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \\ &= \operatorname{curl}\mathbf{F}.\end{aligned}$$

Example 1. If $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$, find $\operatorname{curl}\mathbf{F}$.

Theorem 16.5.1. *If f is a function of three variables that has continuous second-order partial derivatives, then*

$$\operatorname{curl}(\nabla f) = \mathbf{0}.$$

Proof.

$$\begin{aligned}
\operatorname{curl}(\nabla f) = \nabla \times (\nabla f) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} \\
&= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k} \\
&= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}
\end{aligned}$$

by Clairaut's Theorem. □

Example 2. Show that the vector field $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$ is not conservative.

Theorem 16.5.2. *If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl}\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.*

Example 3. (a) Show that

$$\mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$$

is a conservative vector field.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

Definition 16.5.2. If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the divergence of $\mathbf{F}$ is the function of three variables defined by

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Remark 2. In terms of the gradient operator $\nabla = (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j} + (\partial/\partial z)\mathbf{k}$, the divergence of $\mathbf{F}$ can be written symbolically as the dot product of ∇ and $\mathbf{F}$:

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}.$$

Example 4. If $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$, find $\operatorname{div} \mathbf{F}$.

Theorem 16.5.3. *If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then*

$$\operatorname{div} \operatorname{curl} \mathbf{F} = 0.$$

Proof.

$$\begin{aligned}
\operatorname{div} \operatorname{curl} \mathbf{F} &= \nabla \cdot (\nabla \times \mathbf{F}) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} \\
&= 0.
\end{aligned}$$

□

Example 5. Show that the vector field $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$ can't be written as the curl of another vector field, that is, $\mathbf{F} \neq \operatorname{curl} \mathbf{G}$.

Theorem 16.5.4. *Suppose a plane region D, its boundary curve C, and the functions P and Q satisfy the hypotheses of Green's Theorem where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. Then*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA.$$

Proof. Regarding $\mathbf{F}$ as a vector field on $\mathbb{R}^3$ with third component 0, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\, dx + Q\, dy$$

and

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}.$$

Therefore

$$(\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y},$$

and the result follows by Green's Theorem. □

Theorem 16.5.5. *Suppose a plane region D, its boundary curve C, and the functions P and Q satisfy the hypotheses of Green's Theorem where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. Then*

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}(x,y)\,dA.$$

Proof. If C is given by the vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad a \le t \le b$$

then the unit tangent vector is

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{j}$$

and the outward unit normal vector to C is given by

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{j}.$$

Thus

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{n}\,ds &= \int_a^b (\mathbf{F}\cdot\mathbf{n})(t)|\mathbf{r}'(t)|\,dt \\
&= \int_a^b \left[\frac{P(x(t),y(t))y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t),y(t))x'(t)}{|\mathbf{r}'(t)|}\right]|\mathbf{r}'(t)|\,dt \\
&= \int_a^b P(x(t),y(t))y'(t)\,dt - Q(x(t),y(t))\,x'(t)\,dt \\
&= \int_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)dA
\end{aligned}$$

by Green's Theorem. □

16.6 Parametric Surfaces and Their Areas

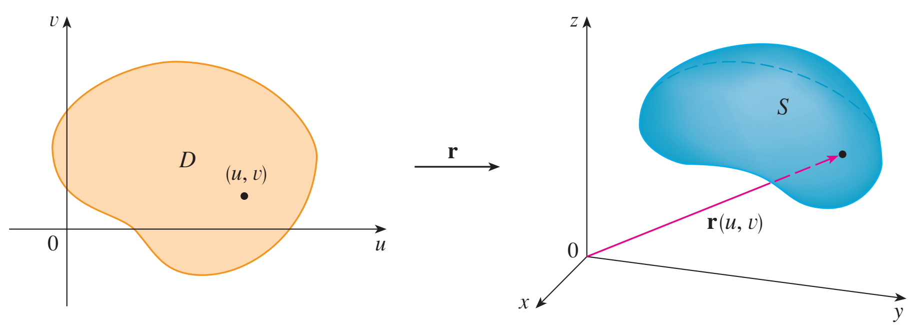
Definition 16.6.1. Suppose that

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

is a vector-valued function defined on a region D in the uv-plane. So x, y, and z, the component functions of $\mathbf{r}$, are functions of the two variables u and v with domain D. The set of all points (x,y,z) in $\mathbb{R}^3$ such that

$$x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v)$$

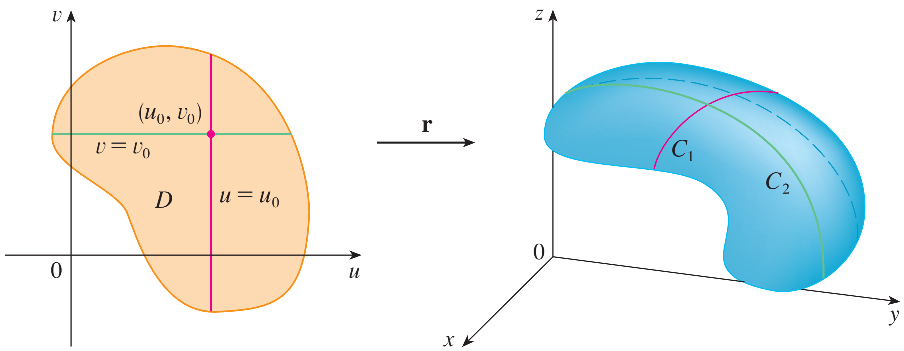
and (u,v) varies throughout D, is called a parametric surface S and the equations are called parametric equations of S. The surface S is traced out by the tip of the position vector $\mathbf{r}(u,v)$ as (u,v) moves throughout the region D. (See the figure.)



Example 1. Identify and sketch the surface with vector equation

$$\mathbf{r}(u,v) = 2\cos u\mathbf{i} + v\mathbf{j} + 2\sin u\mathbf{k}.$$

Definition 16.6.2. If a parametric surface S is given by a vector function $\mathbf{r}(u, v)$ and we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a curve C_1 lying on S. (See the figure.)



Similarly, if we keep v constant by putting $v = v_0$, we get a curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S. We call these curves grid curves.

Example 2. Use a computer algebra system to graph the surface

$$\mathbf{r}(u, v) = \langle (2 + \sin v) \cos u, (2 + \sin v) \sin u, u + \cos v \rangle.$$

Which grid curves have u constant? Which have v constant?

Example 3. Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and that contains two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$.

Example 4. Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2.$$

Example 5. Find a parametric representation for the cylinder

$$x^2 + y^2 = 4 \qquad 0 \leq z \leq 1.$$

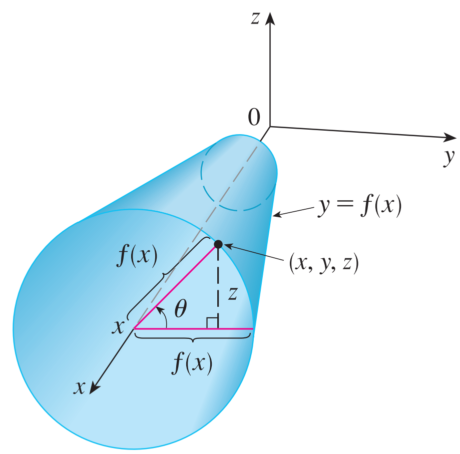
Example 6. Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$.

Example 7. Find a parametric representation for the surface $z = 2\sqrt{x^2 + y^2}$, that is, the top half the cone $z^2 = 4x^2 + 4y^2$.

Remark 1. Surfaces of revolution can be represented parametrically and thus graphed using a computer. For instance, let's consider the surface S obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis, where $f(x) \geq 0$. Let θ be the angle of rotation as shown in the figure. If (x, y, z) is a point on S, then

$$x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta.$$

Therefore we take x and θ as parameters and regard these equations as parametric equations of S. The parameter domain is given by $a \leq x \leq b$, $0 \leq \theta \leq 2\pi$.



Example 8. Find parametric equations for the surface generated by rotating the curve $y = \sin x$, $0 \leq x \leq 2\pi$, about the x-axis. Use these equations to graph the surface of revolution.

Definition 16.6.3. If S is a parametric surface traced out by a vector function

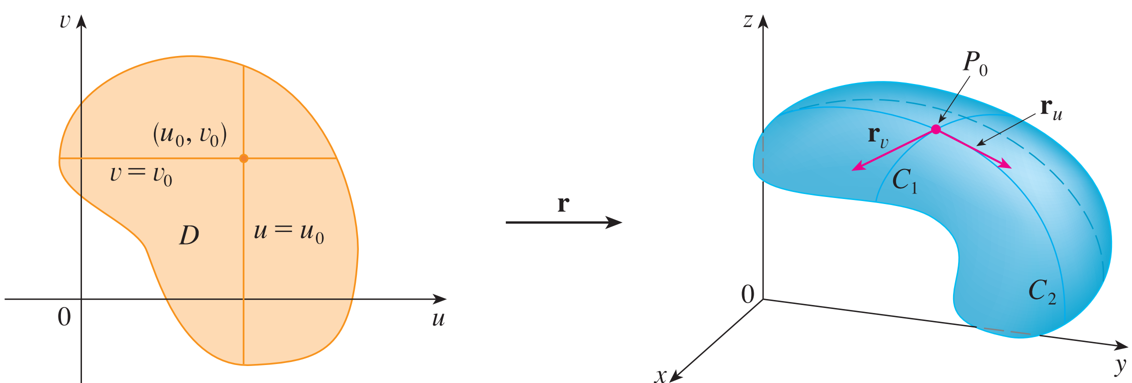
$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$, and if we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a grid curve C_1 lying on S. The tangent vector to C_1 at P_0 is obtained by taking the partial derivative of $\mathbf{r}$ with respect to v:

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$

Similarly, if we keep v constant by putting $v = v_0$, we get a grid curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S, and its tangent vector at P_0 is

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$



If $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$, then the surface S is called smooth (it has no "corners"). For a smooth surface, the tangent plane is the plane that contains the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and the vector $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

Example 9. Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point $(1, 1, 3)$.

Definition 16.6.4. If a smooth parametric surface S is given by the equation

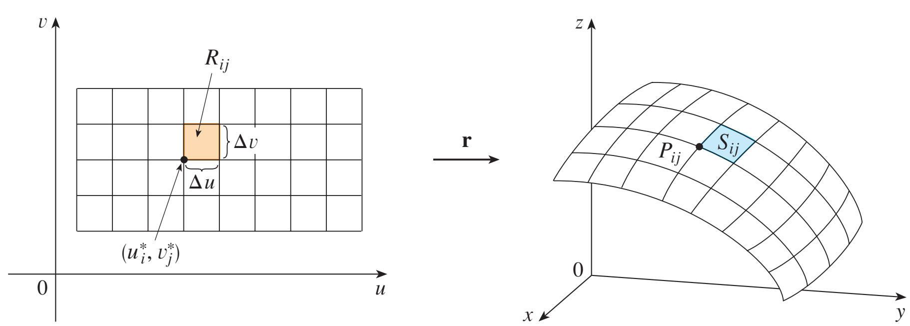
$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the surface area of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$$



Example 10. Find the surface area of a sphere of radius a.

Theorem 16.6.1. *If a surface S has equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives, then the surface areas of S is*

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

Proof. We take x and y as parameters. The parametric equations are

$$x = x \qquad y = y \qquad z = f(x, y)$$

so

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}.$$

Thus we have

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}.$$

□

Example 11. Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

16.7 Surface Integrals

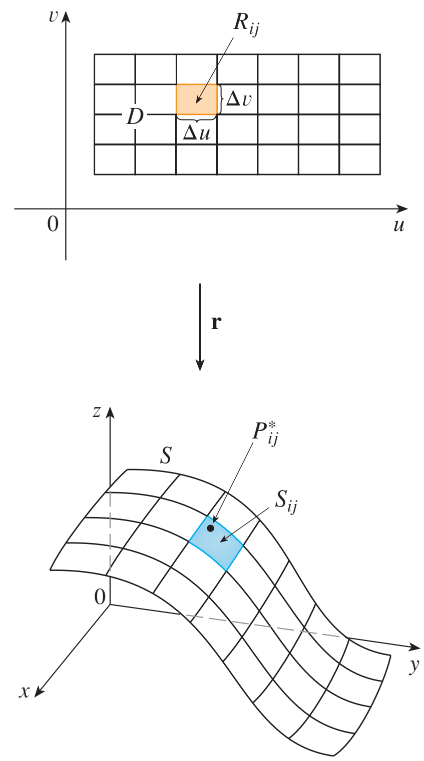
Definition 16.7.1. Suppose that a surface S has a vector equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \qquad (u,v) \in D.$$

Then the surface integral of f over the surface S is

$$\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}$$

where the areas ΔS_{ij} are of patches of S that correspond to subrectangles R_{ij} with dimensions Δu and Δv, and the points P_{ij}^* are sample points in each patch.



Remark 1. It can be shown, even when the parameter domain D is not a rectangle, that

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA,$$

and thus

$$\iint_S 1\,dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA = A(S).$$

Example 1. Compute the surface integral $\iint_S x^2\,dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Theorem 16.7.1. *If S is a surface with equation $z = g(x, y)$, then*

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA.$$

Proof. Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x,y)$$

and so we have

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}.$$

Thus

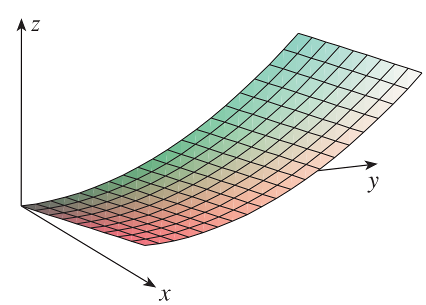
$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}.$$

□

Example 2. Evaluate $\iint_S y\,dS$, where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$. (See the figure.)

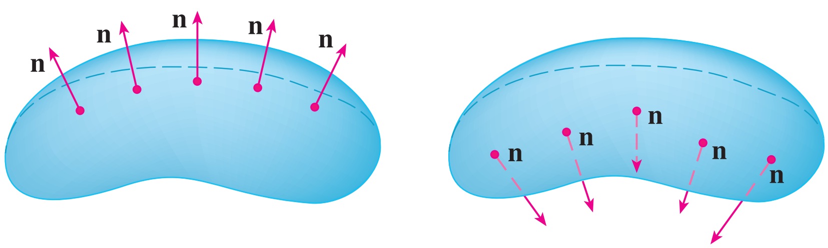


Definition 16.7.2. If S is a piecewise-smooth surface, that is, a finite union of smooth surfaces $S_1, S_2, \ldots, S_n$ that intersect only along their boundaries, then the surface integral of f over S is defined by

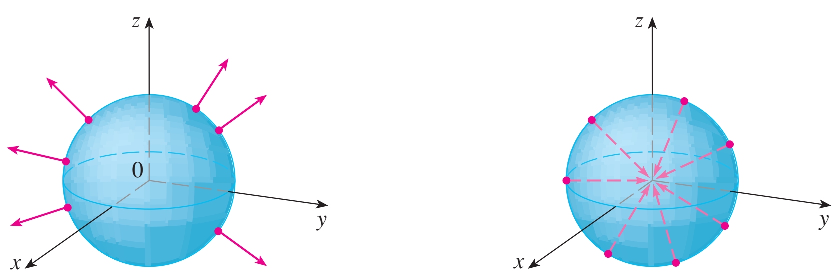
$$\iint_S f(x,y,z)\,dS = \iint_{S_1} f(x,y,z)\,dS + \cdots + \iint_{S_n} f(x,y,z)\,dS.$$

Example 3. Evaluate $\iint_S z\,dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \leq 1$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.

Definition 16.7.3. If S is a surface that has a tangent plane at every point (x, y, z) (except at any boundary point), and if it is possible to choose a unit normal vector $\mathbf{n}$ at every such point so that $\mathbf{n}$ varies continuously over S, then S is called an oriented surface and the given choice of $\mathbf{n}$ provides S with an orientation. There are two possible orientations for any orientable surface (see the figure).



Remark 2. For a closed surface, that is, a surface that is the boundary of a solid region E, the convention is that the positive orientation is the one for which the normal vectors point outward from E, and inward-pointing normals give the negative orientation (see the figure).



Positive orientation Negative Orientation

Definition 16.7.4. If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

This integral is also called the flux of $\mathbf{F}$ across S.

Theorem 16.7.2. *If S is given by a vector function $\mathbf{r}(u, v)$, then*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA$$

where D is the parameter domain.

Proof. If S is given by a vector function $\mathbf{r}(u, v)$, then $\mathbf{n}$ is given by

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

and thus we have

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, dS \\ &= \iint_D \left[\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\right] |\mathbf{r}_u \times \mathbf{r}_v|\, dA.\end{aligned}$$

□

Example 4. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Remark 3. In the case of a surface S given by a graph $z = g(x, y)$, we can think of x and y as parameters and write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \right).$$

Thus

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA.$$

Example 5. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Definition 16.7.5. If $\mathbf{E}$ is an electric field, then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the electric flux of $\mathbf{E}$ through the surface S. Gauss's Law says that the net charge enclosed by a closed surface S is

$$Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

where ε_0 is a constant (called the permittivity of free space) that depends on the units used.

Definition 16.7.6. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then the heat flow is defined as the vector field

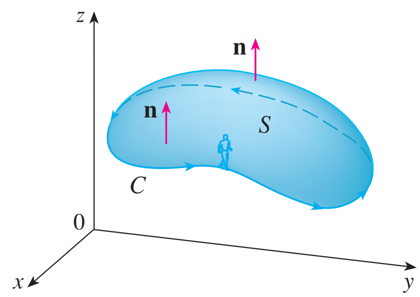
$$\mathbf{F} = -K\nabla u$$

where K is an experimentally determined constant called the conductivity of the substance. The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}.$$

Example 6. The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

16.8 Stokes' Theorem



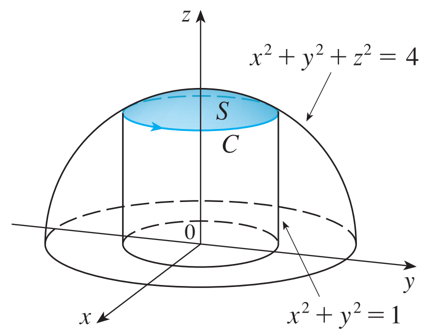
Definition 16.8.1. The figure shows an oriented surface with unit normal vector $\mathbf{n}$. The orientation of S induces the positive orientation of the boundary curve C shown in the figure. This means that if you walk in the positive direction around C with your head pointing in the direction of $\mathbf{n}$, then the surface will always be on your left.

Theorem 16.8.1 (Stokes' Theorem)**.** *Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

Example 1. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above).

Example 2. Use Stokes' Theorem to compute the integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane. (See the figure.)



16.9 The Divergence Theorem

Definition 16.9.1. Regions E that are simultaneously of types 1, 2, and 3 are called simple solid regions. The boundary of E is a closed surface, and we use the convention that the positive orientation is outward; that is, the unit normal vector $\mathbf{n}$ is directed outward from E.

Theorem 16.9.1 (The Divergence Theorem)**.** *Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then*

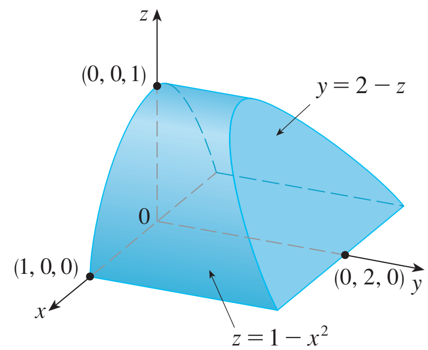
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.$$

Example 1. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

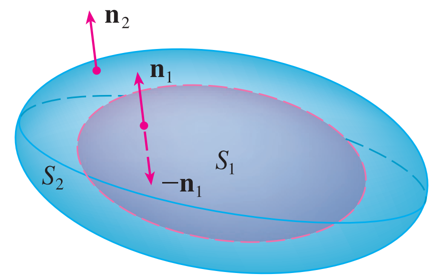
Example 2. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + (y^2 + e^{xz^2})\mathbf{j} + \sin(xy)\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1-x^2$ and the planes $z = 0$, $y = 0$, and $y+z = 2$. (See the figure.)



Remark 1. The Divergence Theorem can be extended to apply to regions that are finite unions of simple solid regions. For example, let's consider the region E that lies between the closed surfaces S_1 and S_2 where S_1 lies inside S_2. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be outward normals of S_1 and S_2. Then the boundary surface of E is $S = S_1 \cup S_2$ and its normal $\mathbf{n}$ is given by $\mathbf{n} = -\mathbf{n}_1$ on S_1 and $\mathbf{n} = \mathbf{n}_2$ on S_2. (See the figure.) Applying the Divergence Theorem to S, we get



$$\begin{aligned}\iiint_E \operatorname{div} \mathbf{F}\, dV &= \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS \\ &= \iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1)\, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS \\ &= -\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.\end{aligned}$$

Example 3. In Example 16.1.5 we considered the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$$

where the electric charge Q is located at the origin and $\mathbf{x} = \langle x, y, z \rangle$ is a position vector. Use the Divergence Theorem to show that the electric flux of $\mathbf{E}$ through any closed surface S_2 that encloses the origin is

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = 4\pi\varepsilon Q.$$

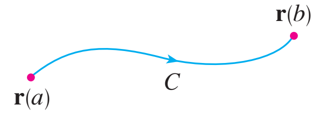
16.10 Summary

Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

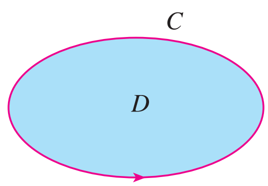
Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



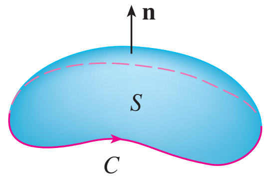
Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$$



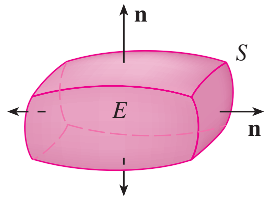
Stokes' Theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$



Chapter 17

Second-Order Differential Equations

17.1 Second-Order Linear Equations

Definition 17.1.1. A second-order linear differential equation has the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

where P, Q, R, and G are continuous functions.

Definition 17.1.2. When $G(x) = 0$, for all x, in the equation in Definition 17.1.1. it is called a homogeneous linear equation. Thus the form of a second-order linear homogeneous differential equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0.$$

If $G(x) \neq 0$ for some x, the equation is nonhomogeneous.

Theorem 17.1.1. *If $y_1(x)$ and $y_2(x)$ are both solutions of a linear homogeneous equation and c_1 and c_2 are any constants, then the linear combination*

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution.

Proof. Since y_1 and y_2 are solutions of a linear homogeneous equation, we have

$$\begin{aligned} P(x)y_1'' + Q(x)y_1' + R(x)y_1 &= 0 \\ P(x)y_2'' + Q(x)y_2' + R(x)y_2 &= 0. \end{aligned}$$

Therefore, using the basic rules for differentiation, we have

$$\begin{aligned} P(x)y''+&Q(x)y' + R(x)y \\ &= P(x)(c_1y_1 + c_2y_2)'' + Q(x)(c_1y_1 + c_2y_2)' + R(x)(c_1y_1 + c_2y_2) \\ &= P(x)(c_1y_1'' + c_2y_2'') + Q(x)(c_1y_1' + c_2y_2') + R(x)(c_1y_1 + c_2y_2) \\ &= c_1[P(x)y_1'' + Q(x)y_1' + R(x)y_1] + c_2[P(x)y_2'' + Q(x)y_2' + R(x)y_2] \\ &= c_1(0) + c_2(0) = 0. \end{aligned}$$

□

Definition 17.1.3. Solutions y_1 and y_2 to a linear homogeneous equation are linearly independent if neither y_1 nor y_2 is a constant multiple of the other. Otherwise, they are linearly dependent.

Theorem 17.1.2. *If y_1 and y_2 are linearly independent solutions of a linear homogeneous equation on an interval, and $P(x)$ is never 0, then the general solution is given by*

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

where c_1 and c_2 are arbitrary constants.

Remark 1. If $y = e^{rx}$ then $y' = re^{rx}$ and $y'' = r^2e^{rx}$, so $y = e^{rx}$ is a solution of

$$ay'' + by' + cy = 0$$

if

$$\begin{aligned} ar^2e^{rx} + bre^{rx} + ce^{rx} &= 0 \\ (ar^2 + br + c)e^{rx} &= 0. \end{aligned}$$

But e^{rx} is never 0. Thus $y = e^{rx}$ is a solution if r is a root of the equation $ar^2 + br + c = 0$, called the auxiliary equation (or characteristic equation) of the differential equation $ay'' + by' + cy = 0$. The roots r_1 and r_2 of the auxiliary equation can be found by factoring or using the quadratic formula:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Theorem 17.1.3 (Case I: $b^2 - 4ac > 0$)**.** *If the roots r_1 and r_2 of the auxiliary equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of $ay'' + by' + cy = 0$ is*

$$y = c_1e^{r_1x} + c_2e^{r_2x}.$$

Proof. In this case the roots r_1 and r_2 of the auxiliary equation are real and distinct, so $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are two linearly independent solutions of $ay'' + by' + cy = 0$. □

Example 1. Solve the equation $y'' + y' - 6y = 0$.

Example 2. Solve $3\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$.

Theorem 17.1.4 (Case II: $b^2 - 4ac = 0$)**.** *If the auxiliary equation $ar^2 + br + c = 0$ has only one real root r, then the general solution of $ay'' + by' + cy = 0$ is*

$$y = c_1 e^{rx} + c_2 x e^{rx}.$$

Proof. By the quadratic formula,

$$r = -\frac{b}{2a} \qquad \text{so} \quad 2ar + b = 0.$$

We know that $y_1 = e^{rx}$ is one solution of $ay'' + by' + cy = 0$. We now verify that $y_2 = xe^{rx}$ is also a solution:

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2re^{rx} + r^2xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx} \\ &= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx} \\ &= 0(e^{rx}) + 0(xe^{rx}) = 0. \end{aligned}$$

Since $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent solutions, Theorem 17.1.2 provides us with the general solution. □

Example 3. Solve the equation $4y'' + 12y' + 9y = 0$.

Theorem 17.1.5 (Case III: $b^2 - 4ac < 0$)**.** *If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is*

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x).$$

Proof. Using Euler's equation

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

we write the solution of the differential equation as

$$\begin{aligned} y &= C_1 e^{r_1 x} + C_2 e^{r_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} \\ &= C_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + C_2 e^{\alpha x}(\cos\beta x - i\sin\beta x) \\ &= e^{\alpha x}[(C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x] \\ &= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) \end{aligned}$$

where $c_1 = C_1 + C_2$, $c_2 = i(C_1 - C_2)$. □

Example 4. Solve the equation $y'' - 6y' + 13y = 0$.

Definition 17.1.4. An initial-value problem for a second-order linear differential equation consists of finding a solution y of the differential equation that also satisfies initial conditions of the form

$$y(x_0) = y_0 \qquad y'(x_0) = y_1$$

where y_0 and y_1 are given constants.

Example 5. Solve the initial-value problem

$$y'' + y' - 6y = 0 \qquad y(0) = 1 \qquad y'(0) = 0.$$

Example 6. Solve the initial-value problem

$$y'' + y = 0 \qquad y(0) = 2 \qquad y'(0) = 3.$$

Definition 17.1.5. A boundary-value problem for a second-order linear differential equation consists of finding a solution y of the differential equation that also satisfies boundary conditions of the form

$$y(x_0) = y_0 \qquad y(x_1) = y_1.$$

Example 7. Solve the boundary problem

$$y'' + 2y' + y = 0 \qquad y(0) = 1 \qquad y(1) = 3.$$

17.2 Nonhomogeneous Linear Equations

Theorem 17.2.1. *The general solution of the nonhomogeneous differential equation* $ay'' + by' + cy = G(x)$ *can be written as*

$$y(x) = y_p(x) + y_c(x)$$

where y_p *is a particular solution of* $ay'' + by' + cy = G(x)$ *and* y_c *is the general solution of the* complementary equation $ay'' + by' + cy = 0$.

Proof. We verify that if y is any solution of $ay'' + by' + cy = G(x)$, then $y - y_p$ is a solution of the complementary equation. Indeed

$$\begin{aligned} a(y - y_p)'' + b(y - y_p)' + c(y - y_p) &= ay'' - ay_p'' + by' - by_p' + cy - cy_p \\ &= (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p) \\ &= G(x) - G(x) = 0. \end{aligned}$$

This shows that every solution is of the form $y(x) = y_p(x) + y_c(x)$. It remains to show that every function of this form is a solution. Indeed

$$\begin{aligned} a(y_p + y_c)'' + b(y_p + y_c)' + c(y_p + y_c) &= ay_p'' + ay_c'' + by_p' + by_c' + cy_p + cy_c \\ &= (ay_p'' + by_p' + cy_p) + (ay_c'' + by_c' + cy_c) \\ &= G(x) + 0 = G(x). \end{aligned}$$

□

Remark 1 (The Method of Undetermined Coefficients).

1. If $G(x) = e^{kx}P(x)$, where P is a polynomial of degree n, then try $y_p(x) = e^{kx}Q(x)$, where $Q(x)$ is an nth-degree polynomial (whose coefficients are determined by substituting in the differential equation).

2. If $G(x) = e^{kx}P(x)\cos mx$ or $G(x) = e^{kx}P(x)\sin mx$, where P is an nth-degree polynomial, then try

$$y_p(x) = e^{kx}Q(x)\cos mx + e^{kx}R(x)\sin mx$$

where Q and R are nth-degree polynomials.

Modification: If any term of y_p is a solution of the complementary equation, multiply y_p by x (or by x^2 if necessary).

Example 1. Solve the equation $y'' + y' - 2y = x^2$.

Example 2. Solve $y'' + 4y = e^{3x}$.

Example 3. Solve $y'' + y' - 2y = \sin x$.

Example 4. Solve $y'' - 4y = xe^x + \cos 2x$.

Example 5. Solve $y'' + y = \sin x$.

Example 6. Determine the form of the trial solution for the differential equation $y'' - 4y' + 13y = e^{2x} \cos 3x$.

Remark 2. Suppose we have already solved the homogeneous equation $ay'' + by' + cy = 0$ and written the solution as

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where y_1 and y_2 are linearly independent solutions. We replace the constants (or parameters) c_1 and c_2 by arbitrary functions $u_1(x)$ and $u_2(x)$. We then look for a particular solution of the nonhomogeneous equation $ay''+by'+cy = G(x)$ of the form

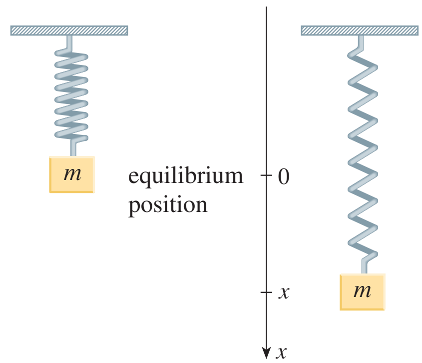
$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

This method is called variation of parameters because we have varied the parameters c_1 and c_2 to make them functions.

Example 7. Solve the equation $y'' + y = \tan x$, $0 < x < \pi/2$.

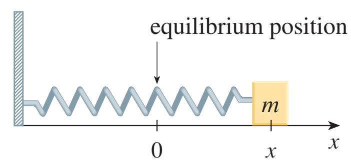
17.3 Applications of Second-Order Differential Equations

Remark 1. Consider the motion of an object with mass m at the end of a spring that is either vertical (as in the first figure) or horizontal on a level surface (as in the second figure). Hooke's Law says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x:



$$\text{restoring force} = -kx$$

where k is a positive constant (called the spring constant). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have



$$m\frac{d^2x}{dt^2} = -kx \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + kx = 0.$$

This is a second-order linear differential equation. Its auxiliary equation is $mr^2 + k = 0$ with roots $r = \pm\omega i$, where $\omega = \sqrt{k/m}$. Thus the general solution is

$$x(t) = c_1 \cos\omega t + c_2 \sin\omega t$$

which can also be written as

$$x(t) = A\cos(\omega t + \delta)$$

where

$$\omega = \sqrt{k/m}$$
$$A = \sqrt{c_1^2 + c_2^2}$$
$$\cos\delta = \frac{c_1}{A} \qquad \sin\delta = -\frac{c_2}{A}.$$

This type of motion is called simple harmonic motion.

Example 1. A spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7 m. If the spring is stretched to a length of 0.7 m and then released with initial velocity 0, find the position of the mass at any time t.

Remark 2. Assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion. Thus

$$\text{damping force} = -c\frac{dx}{dt}$$

where c is a positive constant, called the damping constant. Thus, in this case, Newton's Second Law gives

$$m\frac{d^2x}{dt^2} = \text{restoring force} + \text{damping force} = -kx - c\frac{dx}{dt}$$

or

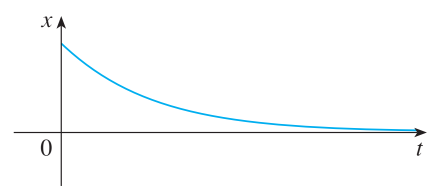
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

This is a second-order linear differential equation and its auxiliary equation is $mr^2 + cr + k = 0$. The roots are

$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \qquad r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}.$$

Case I: $c^2 - 4mk > 0$ (overdamping).
In this case r_1 and r_2 are distinct real roots and

$$x = c_1e^{r_1t} + c_2e^{r_2t}.$$

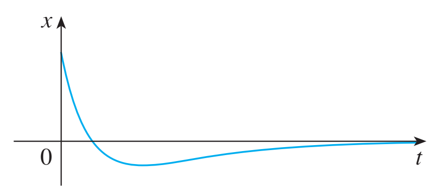


Case II: $c^2 - 4mk = 0$ (critical damping).
This case corresponds to equal roots

$$r_1 = r_2 = -\frac{c}{2m}$$

and the solution is given by

$$x = (c_1 + c_2t)e^{-(c/2m)t}.$$



Case III: $c^2 - 4mk < 0$ (underdamping).
Here the roots are complex:

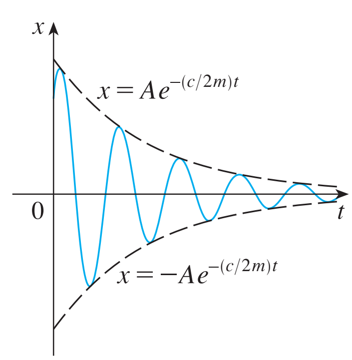
$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = -\frac{c}{2m} \pm \omega i$$

where

$$\omega = \frac{\sqrt{4mk - c^2}}{2m}.$$

The solution is given by

$$x = e^{-(c/2m)t}(c_1\cos\omega t + c_2\sin\omega t).$$



Example 2. Suppose that the spring of Example 1 is immersed in a fluid with damping constant $c = 40$. Find the position of the mass at any time t if it starts from the equilibrium position and is given a push to start it with an initial velocity of 0.6 m/s.

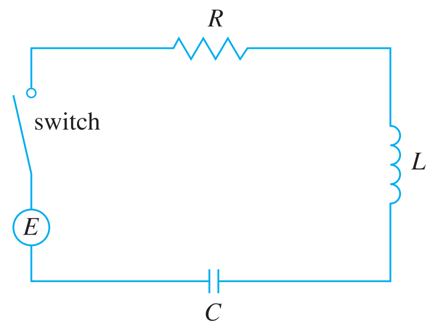
Remark 3. Suppose that, in addition to the restoring force and the damping force, the motion of the spring is affected by an external force $F(t)$. Then Newton's Second Law gives

$$\begin{aligned} m\frac{d^2x}{dt^2} &= \text{restoring force} + \text{damping force} + \text{external force} \\ &= -kx - c\frac{dx}{dt} + F(t). \end{aligned}$$

Thus, instead of the homogeneous equation, the motion of the spring is now governed by the following nonhomogeneous differential equation:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t).$$

Remark 4. The circuit shown in the figure contains an electromotive force E (supplied by a battery or generator), a resistor R, an indicator L, and a capacitor C, in series. If the charge on the capacitor at time t is $Q = Q(t)$, then the current is the rate of change of Q with respect to t: $I = dQ/dt$. It is known from physics that the voltage drops across the resistor, inductor and capacitor are



$$RI \qquad L\frac{dI}{dt} \qquad \frac{Q}{C}$$

respectively. Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage:

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t).$$

Since $I = dQ/dt$, this equation becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

which is a second-order linear differential equation with constant coefficients. If the charge Q_0 and the current I_0 are known at time 0, then we have the initial conditions

$$Q(0) = Q_0 \qquad Q'(0) = I(0) = I_0.$$

A differential equation for the current can be obtained by differentiating with respect to t and remembering that $I = dQ/dt$:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t).$$

Example 3. Find the charge and current at time t in the circuit of the figure if $R = 40\ \Omega$, $L = 1$ H, $C = 16 \times 10^{-4}$ F, $E(t) = 100\cos 10t$, and the initial charge and current are both 0.

17.4 Series Solutions

Example 1. Use power series to solve the equation $y'' + y = 0$.

Example 2. Solve $y'' - 2xy' + y = 0$.

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Bibliography

[1] Stewart, James. *Calculus: Early Transcendentals.* Boston, MA, USA: Cengage Learning, 2016. Print.